Final exam MATH 364

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Problem 1. *Let $\mathcal{I}_{\mathbb{Q}}$ be the set of all open intervals with* rational *endpoints. Prove that this set is* **countable**.

Problem 2. *Let $\sigma : \mathbb{R} \to \mathbb{R}$. Suppose that*

$$\lim_{x\to 0} \frac{\sigma(x)}{x^3} = 3$$

Prove that the function $\sigma(x)$ satisfies

$$\lim_{x\to 0} \sigma(x) = 0.$$

Problem 3. *Using the definition, prove that the following function is differentiable at $x = 0$*

$$f(x) = \begin{cases} 2x + x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Problem 4. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences of real numbers such that*

$$\lim_{n\to\infty} x_n = L$$

and for all $n \in \mathbb{N}$ we also have

$$x_n < y_n < x_{4n} \ .$$

Find the limit of y_n.

Problem 5.

- *Write the definition of a function $f : \mathcal{D} \to \mathbb{R}$ being* **uniformly continuous**
- *Let f, g be uniformly continuous on $\mathbb{R}$; prove that any* **linear combination** *is also uniformly continuous. We recall that a* **linear combination** *is any expression of the form*

  $$af(x) + bg(x) \ ,$$

  for some real constants a, b.
- *Show that $f(x) = \frac{1}{x-2}$ is not uniformly continuous on $(2, 3]$.*

Problem 6. *Show that the equation*

$$\mathrm{e}^x = 3x^2$$

has at least two positive solutions.

Problem 7. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable and such that*

$$f''(x) + f'(x) > 0 \ , \quad \forall x \in (a, b)$$

Show that

$$f(x) \leq \max\{f(a), f(b)\} \ , \qquad \forall x \in [a, b].$$

You can use the following
Criterion *Let $f(x) : I \to \mathbb{R}$ be twice differentiable and let $c \in I$ be an interior point such that*

$$f'(c) = 0$$

If $f''(c) > 0$ then c is a point of **local minimum***; if $f''(c) < 0$ then c is a point of* **local maximum**.

Problem 8. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\forall x, y \in [0,1]$*

$$|f(x) - f(y)| < \frac{1}{2}|x - y|$$

Consider the sequence

$$x_0 := 1 \ , \quad x_n = f(x_{n-1}) \ , \quad n \geq 1$$

1. *State –in general– the definition of Cauchy property for a sequence;*
2. *Show that the sequence defined above has the Cauchy property;*
3. *Deduce that the limit $\lim_{n\to\infty} x_n = L$ exists and that $f(L) = L$.*

Hint for point 2: *prove –by induction– that*

$$|x_{n+1} - x_n| < \frac{1}{2^n}|x_1 - x_0|$$

and deduce from this the Cauchy property.

Problem 9. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows*

$$f(x) = \begin{cases} 2x + 3 & x \in \mathbb{Q} \\ 3x - 4 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Find all points where the function is continuous, if any, and prove the continuity there.

Problem 10. *A function $f : I \to \mathbb{R}$ is said to be α–Hölder for $\alpha > 0$ if*

$$\exists C \geq 0 : \ \forall x, y \in I, \quad |f(x) - f(y)| < C|x - y|^{\alpha}$$

Prove that

1. *an α–Hölder function is continuous;*
2. *the function $f : [0,1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is $1/2$-Hölder (i.e. with $\alpha = \frac{1}{2}$ in the definition above);*
3. *the same function as above is* **not** *1–Hölder (i.e. with $\alpha = 1$ in the definition above).*

[Hint for the last point: start by writing the negation of the definition for $\alpha = 1$ and consider $y = 0$.]

Problem 11. (a) *Write the definition of* $\limsup$ *for a sequence of numbers.*
(b) *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

$$\forall R > 3 \ \exists N \in \mathbb{N} : \ \forall n \geq N \qquad x_n \leq R$$

Prove that

$$\limsup_{n\to\infty} x_n \leq 3$$