1, 2, 8 — ме

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	364	All
Examination	**Date**	**Pages**
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Instructors		**Course Examiner**
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Special Instructions:

▷ Calculators are **not** permitted.

MARKS

[7] **1.** (a) Let $A, B \subset \mathbb{R}$, $A \cap B = \emptyset$ and $\sup A = \inf B$.
Prove that $\forall\, \varepsilon > 0$, $\exists\ a \in A, b \in B$ such that $b - a < \varepsilon$.

[3] (b) Give an example of two sets $A, B \subset \mathbb{R}$ such that $\inf A = \inf B$ and $\sup A = \sup B$, but $A \cap B = \emptyset$.

[7] **2.** (a) Prove that the set $S = \{x = a + b\sqrt{2} : a \in \mathbb{Q}, b \in \mathbb{Z}\}$ is countable.

[6] (b) Prove that the set of irrational numbers is uncountable (you may use the following facts: the set of rational numbers is countable and the set of real numbers is uncountable).

[5] **3.** (a) Let the sequence $(a_n)_{n\in\mathbb{N}}$ converge to zero and the sequence $(b_n)_{n\in\mathbb{N}}$ be bounded. Prove that the sequence $(c_n)_{n\in\mathbb{N}}$, where $c_n = a_n b_n$, also converges to zero.

[3] (b) What can you say about the sequence $(c_n)_{n\in\mathbb{N}}$ above if $\lim\limits_{n\to\infty} a_n = a \neq 0$? Explain.

[5] (c) Use the Squeeze Theorem to find the limit of the sequence

$$b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

[3] **4.** (a) State the definition of a monotone sequence.

[3] (b) What can you say about the set E of all subsequential limits of a monotone sequence? Explain.

[6] (c) Prove that if a sequence $(a_n)_{n\in\mathbb{N}}$ is not bounded above, then it has an increasing subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} a_{n_k} = +\infty$.

[4] (d) Find the limit superior and the limit inferior of the sequence

$$a_n = \frac{1+(-1)^n}{2} + \frac{1}{2n^2}$$

[6] **5.** (a) State the definition of the limit of a function f at a point a

i) using the $\varepsilon - \delta$ formulation;

ii) using sequences.

[5] (b) Use either of the definitions above to prove that if $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} g(x) = L_2$, then

$$\lim_{x\to a}\,[2f(x) + 5g(x)] = 2L_1 + 5L_2$$

[6] **6.** (a) Let a function f be continuous at a point c and $f(c) = 1$. Prove that there exists a neighbourhood U of the point c such that $f(x) > \frac{1}{2}$, $\forall\, x \in U$.

[4] (b) Consider the function $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ \frac{1}{x^2} & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

Can you find real number(s) where f is continuous? Explain.

[7] (c) Let f and g be continuous functions on $[a,b]$ such that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in (a,b)$.
(You may use the Intermediate Value Theorem.)

7. Use the definition of uniform continuity to prove that for any $a > 0$,

[4] (a) $f(x) = x^2$ is uniformly continuous on $(0, a]$.

[4] (b) $f(x) = x^{-2}$ is not uniformly continuous on $(0, a]$.

[7] **8.** (a) Prove directly from the definition that the function $f(x) = x|x|$ is differentiable on $\mathbb{R}$. Find $f'(x)$. What can you say about $f''(x)$?

[5] (b) Use the Mean Value Theorem to prove the inequality $e^x > 1 + x\,,\ \forall\, x > 0$.

[5] **Bonus**

Prove that the two definitions i) and ii) of the limit of a function f at a point a (question 5(a)) are equivalent.