Question 1

(a) (i)

$$\lim_{x\to-3}\frac{x^2-3x+2}{x-1}=\frac{(-3)^2-3(-3)+2}{-3-1}=-\frac{20}{4}=-5$$

(ii)

$$\lim_{x\to5+}\frac{x^2-16}{x-5}=\infty$$
$$\lim_{x\to5-}\frac{x^2-16}{x-5}=-\infty$$

Thus, two-sided limit does not exist.

(iii)

$$\lim_{x\to\infty}\frac{-5x^7+3x^2+2}{4-x^2}=\lim_{x\to\infty}\frac{-\frac{5x^7}{x^7}+\frac{3x^2}{x^7}+\frac{2}{x^7}}{\frac{4}{x^7}-\frac{x^2}{x^7}}=\lim_{x\to\infty}\frac{-5+\frac{3}{x^5}+\frac{2}{x^7}}{\frac{4}{x^7}-\frac{1}{x^5}}=\infty$$

(b) (i)

$$\lim_{x\to3}-3g(x)=-3\lim_{x\to3}g(x)=(-3)4=-12$$

(ii)

$$\lim_{x\to3}\sqrt{g(x)}=\sqrt{\lim_{x\to3}g(x)}=\sqrt{4}=2$$

(iii)

$$\lim_{x\to3}\frac{g(x)}{2f(x)}=\frac{\lim_{x\to3}g(x)}{2\lim_{x\to3}f(x)}=\frac{4}{-10}=-\frac{2}{5}$$

(iv)

$$\begin{aligned}\lim_{h\to0}\frac{(x-h)^2-x^2}{h}&=\lim_{h\to0}\frac{x^2-2xh+h^2-x^2}{h}\\&=\lim_{h\to0}\frac{-2xh+h^2}{h}\\&=\lim_{h\to0}\frac{h(-2x+h)}{h}\\&=\lim_{h\to0}(-2x+h)\\&=-2x\end{aligned}$$

Question 2

(a)

$$f'(x) = (-6)10x^9 + (-4)3x^2 = -60x^9 - 12x^2$$

(b)

$$\begin{aligned} f'(x) &= \frac{(2x+5)(e^x-7) - e^x(x^2+5x)}{(e^x-7)^2} \\ &= \frac{2xe^x - 14x + 5e^x - 35 - e^x x^2 - 5xe^x}{(e^x-7)^2} \\ &= \frac{-x^2e^x - 3xe^x - 14x + 5e^x - 35}{(e^x-7)^2} \end{aligned}$$

(c)

$$\frac{dy}{dx} = \frac{2(x^2+3x)(2x+3)}{(x^2+3x)^2} = \frac{4x+6}{x^2+3x}$$

(d)

$$\frac{dy}{dx} = \frac{1}{5}(x+5)^{1/5-1} = \frac{1}{5}(x+5)^{-4/5} = \frac{1}{5\sqrt[5]{(x+5)^4}}$$

(e) Differentiate both sides of the equation with respect to x and use the Chain rule and the Product rule where appropriate:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(e^y - 2)$$

$$\frac{d}{dx}(x)y + x\frac{dy}{dx} = e^y\frac{dy}{dx}$$

$$\frac{dy}{dx}(e^y - x) = y$$

$$\frac{dy}{dx} = \frac{y}{e^y - x}$$

Question 3

The price-demand equation is:

$$x = f(p) = 7000 - 500p$$

Find the price elasticity of demand:

$$\begin{aligned} E(p) &= -\frac{pf'(p)}{f(p)} \\ &= \frac{500p}{7000-500p}E(p) \\ &= \frac{500p}{7000-500p} \leq 1 \end{aligned}$$

Hence,

$$\begin{aligned} 500p &\leq 7000 - 500p \\ p &\leq 7 \end{aligned}$$

Therefore the demand will remain inelastic while the price for a pair of sunglasses is less or equal to 7. Thus an increase in price will lead to an increase in revenue.

Question 4

Let $f(x) = x^2 - x$ and $g(x) = 2x$.

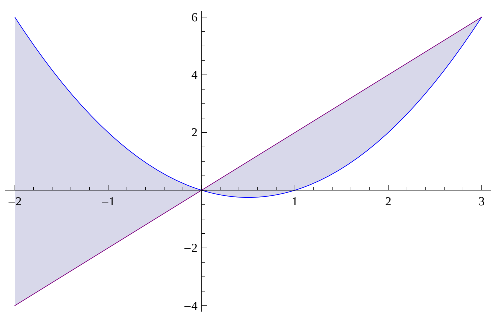

Figure 1: Plot of the parabola $x^2 - x$ and the line $2x$

First find the points of intersection of the two graphs:

$$\begin{aligned} x^2 - x &= 2x \\ x^2 - 3x &= 0 \\ x = 0, &\quad x = 3 \end{aligned}$$

On $[-2,\ 0]$, $f(x) > g(x)$. On $[0,\ 3]$, $g(x) > f(x)$. The area bounded by $f(x)$ and $g(x)$ is:

$$\begin{aligned}\int_{-2}^{0}[f(x)-g(x)]\ \mathrm{d}x+\int_{0}^{3}[g(x)-f(x)]\ \mathrm{d}x &= \int_{-2}^{0}(x^2-3x)dx+\int_{0}^{3}(3x-x^2)\,\mathrm{d}x \\ &= \left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_{-2}^{0}+\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_{0}^{3} \\ &= \frac{8}{3}+\frac{12}{2}+\frac{27}{2}-\frac{27}{3} \\ &= \frac{79}{6}\end{aligned}$$

Question 5

$$f(x) = x^4 - 2x^3$$

Find the x-intercepts:

$$\begin{aligned} x^3(x-2) &= 0 \\ x = 0, \quad x &= 2 \end{aligned}$$

Find the critical points:

$$\begin{aligned} f'(x) = 4x^3 - 6x^2 &= 0 \\ 2x^2(2x-3) &= 0 \\ x = 0, \quad x &= \frac{3}{2} \end{aligned}$$

Find the points of inflection:

$$\begin{aligned} f''(x) = 12x^2 - 12x &= 0 \\ 12x(x-1) &= 0 \\ x = 0, \quad x &= 1 \end{aligned}$$

	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 3/2)$	3/2	$(3/2, \infty)$
$f'(x)$	< 0	0	< 0	< 0	< 0	0	> 0
$f'(x)$	> 0	0	< 0	0	> 0	> 0	> 0
$f'(x)$	Decreases, concave upward	Point of inflection	Decreases, concave downward	Point of inflection	Decreases, concave upward	Absolute minimum	Increases, concave downward

The graph of the function is:

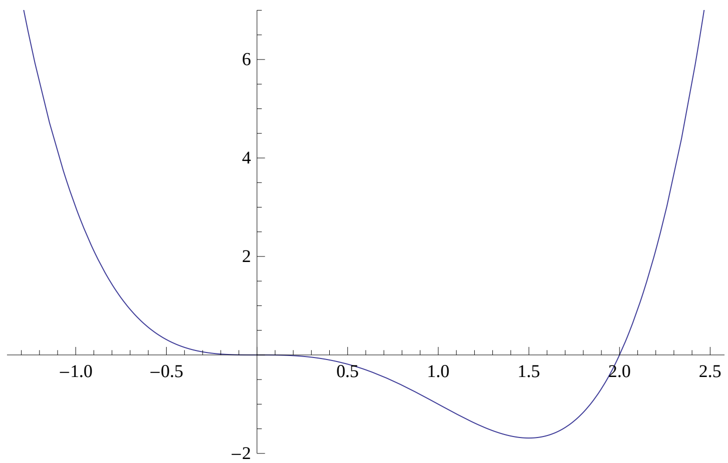

Figure 2: Plot of $x^4 - 2x^3$

Question 6

(a)

$$p = \frac{1000 - x}{20} = 50 - \frac{x}{20}, \quad 0 < x < 1000$$

(b)

$$R(x) = xp(x) = 50x - \frac{x^2}{20}, \quad 0 < x < 1000$$

(c) The marginal revenue at a production level of 400 units is:

$$R'(x) = 50 - \frac{x}{10}$$

$$R'(400) = 50 - 40 = 10$$

Therefore revenue is increasing at a rate of \$10 per unit.

(d) The marginal revenue at a production level of 650 units is:

$$R'(650) = 50 - 65 = -15$$

Therefore revenue is decreasing at a rate of \$15 per unit.

Question 7

$P = 500$ pounds/sq. inch, $T = 250$ Kelvin, $dT/dt = 3$ Kelvin/hour

$$k = \frac{P}{T} = \frac{500}{250} = 2$$

Since $P = 2T$, the rate of change of pressure with respect to time is:

$$\frac{dP}{dt} = 2\frac{dT}{dt}$$

$$\frac{dP}{dt} = 2 \cdot 3 = 6\text{pounds/(sq. inch} \cdot \text{hour)}$$

Question 8

(a)

$$\int e^{5x}\,\mathrm{d}x = \frac{1}{5}e^{5x} + C$$

(b) Let $u = x - 7$. Then $du = dx$ and $x = u + 7$.

$$\begin{aligned}
\int \frac{x}{\sqrt{x-7}}\,\mathrm{d}x &= \int \frac{u+7}{\sqrt{u}}\,\mathrm{d}u \\
&= \int \left(\sqrt{u} + \frac{7}{\sqrt{u}}\right)\mathrm{d}u \\
&= \frac{2u^{3/2}}{3} + 14\ u^{1/2} \\
&= \frac{2}{3}(x-7)^{3/2} + 14(x-7)^{1/2} + C \\
&= \sqrt{x-7}\left(\frac{2}{3}x - \frac{14}{3} + 14\right) + C \\
&= \frac{2}{3}\sqrt{x-7}(x+14) + C
\end{aligned}$$

(c)

$$\int (3x^2 + 5x)\,\mathrm{d}x = x^3 + \frac{5}{2}x^2 + C$$

(d) Let $u = 4 + x^3$. Then $du = 3x^2$ or $1/3\ du = x^2 dx$.

$$\int \frac{x^2}{4 + x^3}\,\mathrm{d}x = \int \frac{1}{3u}\,\mathrm{d}u = \frac{1}{3}\ln|u| = \frac{1}{3}\ln\left|4 + x^3\right| + C$$

(e) Let $u = x^2 + 1$. Then $du = 2x\ dx$ or $1/2\ du = x\ dx$.

$$\int \left(x^2 + 1\right)^{12} x\ dx = \int \frac{u^{12}}{2} du = \frac{u^{13}}{26} = \frac{\left(x^2 + 1\right)^{13}}{26} + C$$

Question 9

(a)

$$\int_0^5 (t^2 - 4)\,\mathrm{d}t = \left[\frac{t^3}{3} - 4t\right]_0^5 = \frac{125}{3} - 20 = \frac{65}{3}$$

(b) Let $u = h^2$. Then $1/2\ du = h\ dh$. The limits of integration become 4 and 9.

$$\int_2^3 e^{h^2} h\,\mathrm{d}h = \int_4^9 \frac{e^u}{2}\,\mathrm{d}u = \left[\frac{e^u}{2}\right]_4^9 = \frac{1}{2}e^4(e^5 - 1) \approx 4024.24$$

Question 10

(a) $R = 5$ cm, $dR = 0.1$ cm

$$V = \frac{4}{3}\pi R^3$$

$$dV = 4\pi R^2 dR$$

$$dV = 4\pi(5)^2(0.1) = 10\pi$$

Therefore the volume of the ice coating is 10π.

(b) Let $f(x) = |x|$. Then $f(x)$ is continuous at 0 and

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - x}{h}$$

We are interested in $f'(0)$:

$$f'(0) = \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Since

$$\lim_{h\to 0+} \frac{|h|}{h} = 1$$
$$\lim_{h\to 0-} \frac{|h|}{h} = -1$$

Two-sided limit does not exist. Hence $f(x) = |x|$ is not differentiable at 0.