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Applied Ordinary Differential Equations
ENGR 213 - Section F
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Exam I (A)

(1) (10 points) Solve the initial value problem

$$x\frac{dy}{dx} + (x-4)y = x^5, \quad y(1) = 1.$$

Solution: An integrating factor is

$$\mu(x) = e^{\int \left(1-\frac{4}{x}\right)dx} = e^{x-4\ln|x|} = e^x \cdot e^{\ln x^{-4}} = e^x \cdot x^{-4} = \frac{e^x}{x^4}.$$

Multiplying both sides of the ODE with this integrating factor leads to

$$\left(\frac{e^x}{x^4}\,y\right)' = e^x.$$

Integrating both sides with respect to x, we obtain

$$\frac{e^x}{x^4}\,y = e^x + C \;\Rightarrow\; y = x^4 + C\frac{x^4}{e^x}, \quad C = \text{constant}.$$

We use $y(1) = 1$ to find the constant C as $y(1) = 1 + C/e = 1 \;\Rightarrow\; C = 0$, thus the solution to the IVP is $y(x) = x^4,\ x > 0$.

(2) (10 points) Solve the given homogeneous equation by the appropriate substitution

$$\frac{dy}{dx} = \frac{3xy}{3x^2+y^2}.$$

You may leave the solution in implicit form.

Solution: We use the standard substitution for homogeneous ODEs of first order, namely $u = y/x$ or $y = ux$ and, thus, $\frac{dy}{dx} = x\,\frac{du}{dx} + u$.

Therefore the ODE becomes

$$x\,\frac{du}{dx} + u = \frac{3u}{3+u^2} \;\Rightarrow\; x\,\frac{du}{dx} = -\frac{u^3}{3+u^2}.$$

The latter is a separable ODE $-\frac{3+u^2}{u^3}\,du = \frac{1}{x}\,dx$, or $-\left(\frac{3}{u^3} + \frac{u^2}{u^3}\right)du = \frac{1}{x}\,dx$. Integrating, we obtain $\frac{3}{2}u^{-2} - \ln|u| = \ln|x| + C$, where C is an arbitrary constant.

Hence we obtain the solution $y(x)$, in implicit form, of the original ODE

$$\frac{3x^2}{2y^2} - \ln|y/x| = \ln|x| + C, \quad C = \text{constant}.$$

(3) (10 points) Solve the exact ordinary differential equation

$$(3 - 2\sin(2x + 3y))\, dx + (2 - 3\sin(2x + 3y))\, dy = 0,$$

leaving the solution in implicit form.

Solution: We seek a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = 3 - 2\sin(2x + 3y), \quad \frac{\partial \phi}{\partial y} = 2 - 3\sin(2x + 3y).$$

Integrating the first equation with respect to x, we obtain $\phi(x, y) = 3x + \cos(2x + 3y) + c(y)$. Consequently,

$$\frac{\partial \phi}{\partial y} = -3\sin(2x + 3y) + c'(y) \quad \Rightarrow \quad c'(y) = 2 \;\; c(y) = 2y + C, \; C = \text{constant}.$$

It follows that the solution to the exact ODE (given in implicit form) is

$$3x + \cos(2x + 3y) + 2y = \tilde{C}, \; \tilde{C} = \text{constant}.$$

(4) (10 points) A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 minutes the temperature of the cake is 140°F. When will it be 100°F?

Solution: Newton's model of cooling tells us that the ODE describing the temperature of the cake $T(t)$ is

$$\frac{dT}{dt} = -K(T - 70), \; T(0) = 210, \; T(30) = 140, \quad K = \text{a constant to be determined}.$$

We solve first for the general solution of the ODE (which we treat as separable):

$$\frac{1}{T - 70}\, dT = -K\, dt \quad \Rightarrow \quad \ln|T - 70| = -Kt + C, \; C = \text{constant}.$$

Using $T(0) = 210$ and $T(30) = 140$, we obtain $C = \ln 140$ and $\ln 70 = -30K + \ln 140 \;\Rightarrow\; 30K = \ln 2 \;\Rightarrow\; K = (\ln 2)/30$.

Thus

$$T = 70 + e^{-Kt+C} = 70 + e^{-t(\ln 2)/30 + \ln 140} = 70 + 140\left(\frac{1}{2}\right)^{t/30}$$

and

$$100 = 70 + 140\left(\frac{1}{2}\right)^{t/30} \quad \Rightarrow \quad t = 30\ln(3/14)/\ln(1/2) = 30\ln(14/3)/\ln 2 \approx 66.67\,\text{min}.$$