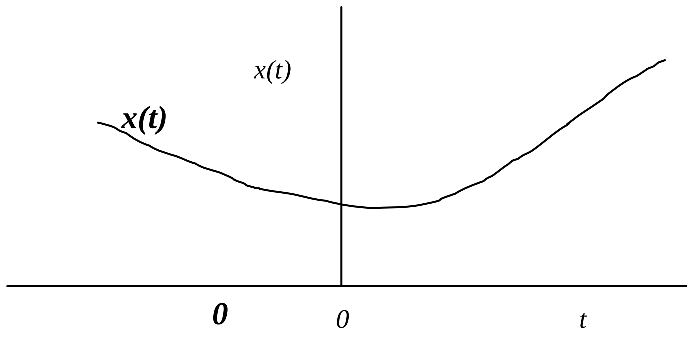
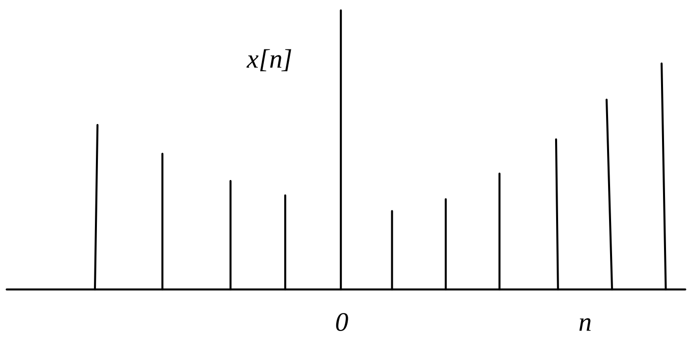
Continuous-Time & Discrete-Time Signals

Continuous-Time:



(a)

Discrete-Time:



(b)

Signal Energy & Power:

• Signal *x(t)* and *x[n]* -- *associated with energy & power:*

• **Energy E & Power P:**

Continuous -time :

$$E = \int_{t_1}^{t_2} \left| x(t)^2 \right| dt$$

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$$

Discrete-time:

$$E = \sum_{N1}^{N2} |x[n]|^2$$

$$P = \frac{1}{N2 - N1} \sum_{N1}^{N2} |x[n]|^2$$

For infinite time interval:

Continuous-time:

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt \, ; \qquad P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Discrete-time:

$$E_\infty = \sum_{-\infty}^{\infty} |x[n]|^2 \, ; \qquad P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$$

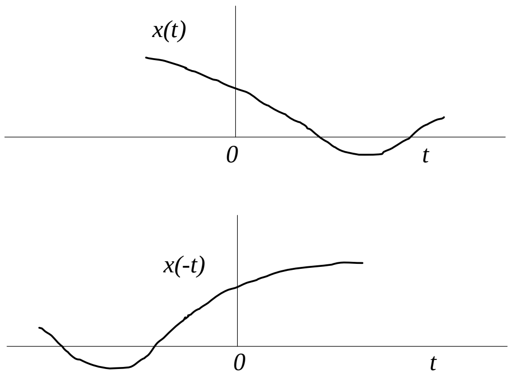
Transformation of Independent Variables (t, and n)

***Time Reversal*:** $t \rightarrow -t$; e.g., $x(t) \rightarrow x(-t)$.

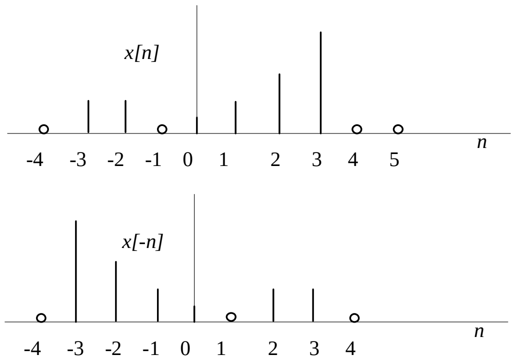
$n \rightarrow -n$; e.g., $x[n] \rightarrow x[-n]$.

Continuous Signal:

• Signal is *reflected* at origin. Example:

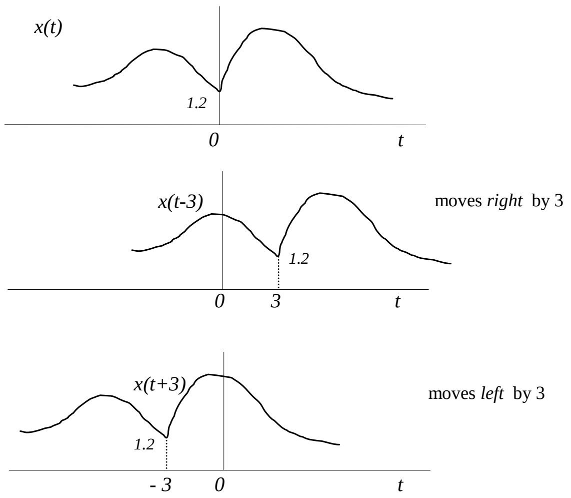


Discrete Signal:

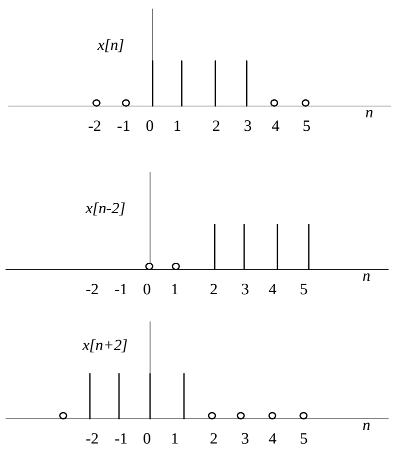


Time Shift:

Continuous Signal: e.g.. $x(t) \rightarrow x(t-3)$, $x(t) \rightarrow x(t+3)$.



Discrete Signal: e.g.. $x[n] \rightarrow x[n-2]$, $x[n] \rightarrow x[n+2]$.



Even & Odd Signal:

- *Even signal:* Continuous: $x(t) = x(-t)$
  Discrete: $x[n] = x[-n]$.
- *Odd Signal:* Continuous: $x(t) = - x(-t)$
  Discrete: $x[n] = - x[-n]$.

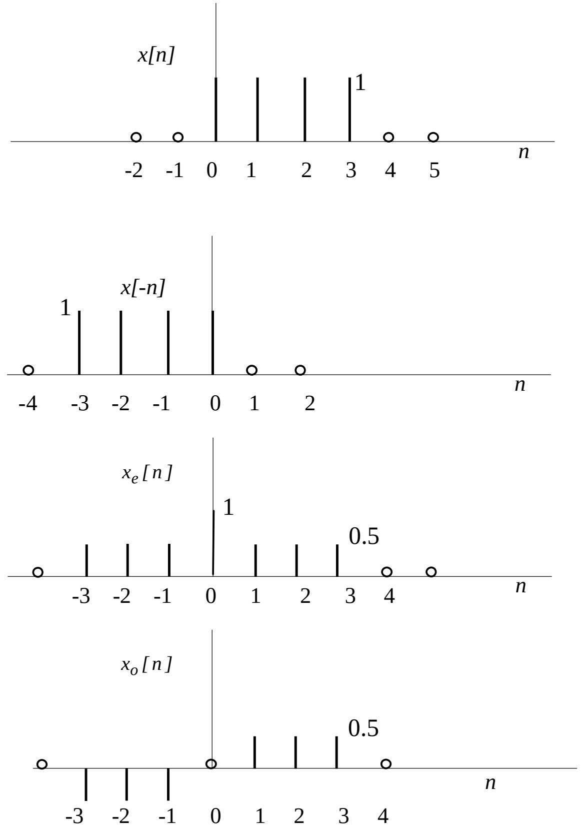
For continuous function $x(t) = x_e(t) + x_o(t)$, giving:

$$x_e(t) = \frac{x(t) + x(-t)}{2}, \text{ and } \quad x_o(t) = \frac{x(t) - x(-t)}{2}$$

For a discrete function $x[n] = x_e[n] + x_o[n]$

$$x_e[n] = \frac{x[n] + x[-n]}{2}, \qquad x_o[n] = \frac{x[n] - x[-n]}{2}$$

Example: Given x{n] below, find $x_e[n]$ and $x_0[n]$



Periodic Signal:

- To be periodic: $x(t) = x(t+T)$, T = period.

Fundamental period T_o= smallest T satisfying the above.

- $x[n] = x[n+N]$, N = period.

Fundamental period N_o= smallest N satisfying the above.

Continuous Periodic Signal:

$$x(t) = e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

With a period T, the signal $x(t) = e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} . e^{j\omega_0 T}$

or, $e^{j\omega_0 T} = 1 = e^{j2\pi k}$, giving $T = \frac{2\pi}{\omega_0} k$

Smallest value of k is 1. Hence,

Fundamental period. $T_0 = \frac{2\pi}{\omega_0}$ $k = \pm 2, \pm 3, ..$ are harmonics.

•For a signal with different component fundamental periods,

Fundamental period = LCM of all fundamental frequencies.

Example: (1.10, p.58)

Find fundamental period of x(t) = 2 cos (10t+1) – sin (4t -1).

2 cos(10t+1) : fundamental period: $\frac{2\pi}{10} = \frac{\pi}{5}$

Sin(4t -1): fundamental period: $\frac{2\pi}{4} = \frac{\pi}{2}$

LCM = π = Fundamental period of *x(t)*.

Discrete Time Signal:

• Complex exponential function: $x[n] = C\,\alpha^n$

For sinusoidal:

$$x[n] = e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

Examples of such functions: (Fig. 1.25, p.25 of text).:

Note that part (c) is not periodic.

Discrete Time Signal (Sinusoidal):

As n is interger: $e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}.e^{j2\pi n} = e^{j\omega_0 n}$

i.e., exponential at ω_0 = exponential at $\omega_0 + 2\pi$

•We need to consider a discrete time signal only in the frequency interval of 2π only.

Condition of periodicity (discrete time signals):

With a fundamental period N,

$$e^{j\omega_0(n+N)} = e^{i\omega_0 n}; \quad \text{or} \quad e^{j\omega_0 N} = 1 = e^{j2\pi m}$$

Which yields: $\frac{\omega_0}{2\pi} = \frac{m}{N}$. This ratio is to be rational for periodicity.

Example: (Prob. 1.26 p.61). (Find fundamental period of:)

(a) $$x[n] = \sin(\frac{6\pi}{7}n + 1).$$

$$\omega_0 = \frac{6\pi}{7}; \quad \frac{\omega_0}{2\pi} = \frac{3}{7}. \qquad N = 7.$$

(b) $$x[n] = \cos(\frac{n}{8} - \pi)$$

$$\omega_0 = \frac{1}{8}; \qquad \frac{\omega_0}{2\pi} = \frac{1}{16\pi} \qquad \rightarrow \text{irrational. Not periodic.}$$

(c) $$x[n] = \cos(\frac{\pi}{2}n)\ \cos(\frac{\pi}{4}n)$$

$$= \frac{1}{2}(\cos\frac{3\pi}{4}n + \cos\frac{\pi}{4}n);$$

$$\cos\frac{3\pi}{4}n \rightarrow \frac{\omega_0}{2\pi} = \frac{3}{8}; \quad \cos\frac{\pi}{4}n \rightarrow \frac{\omega_0}{2\pi} = \frac{1}{8}$$

Hence, N = 8.

(e)

$$x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}k + \frac{\pi}{6}).$$

$$\cos(\frac{\pi}{4}n) \rightarrow \frac{\omega_0}{2\pi} = \frac{1}{8}, \quad N_0 = 8;$$

$$\sin(\frac{\pi}{8}n) \rightarrow \frac{\omega_0}{2\pi} = \frac{1}{16}, \quad N_0 = 16;$$

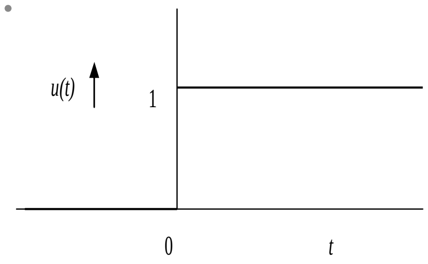
$$\cos(\frac{\pi}{2}k + \frac{\pi}{6}) \rightarrow \frac{\omega_0}{2\pi} = \frac{1}{4}, \quad N_0 = 4;$$

Fundamental period= N = 16 (LCM).

Unit Impulse and Step Functions:

Continuous-time Functions:

Unit Step Function u(t):



$u(t) = 0 \qquad t < 0$

$\quad\;\; = 1 \qquad t > 0.$

Discontinuous at $t = 0$.

Unit Impulse Function δ(t):

- Defined as
$$\delta(t) = \frac{d}{dt} u(t) \quad .. (1)$$

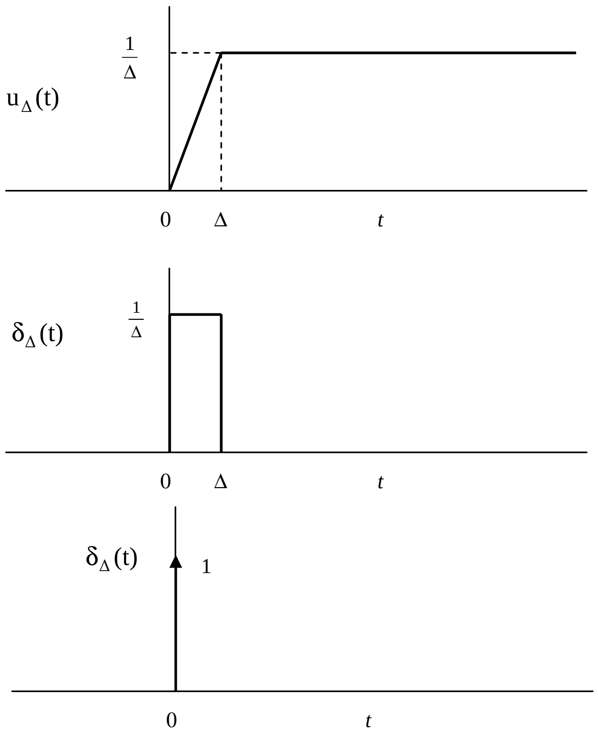
$$u(t) = \mathrm{Lim}_{\Delta \to 0}\, u_{\Delta}(t)$$

$$\delta_{\Delta}(t) = \frac{d}{dt} u_{\Delta}(t)$$

$$\delta(t) = \mathrm{Lim}_{\Delta \to 0}\, \delta_{\Delta}(t)$$

$\delta(t) = 1, \quad t = 0$

$\quad\;\; = 0, \quad$ elsewhere.



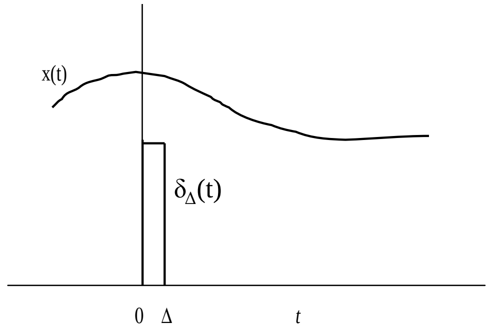
From equation (1):

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \quad .. (2)$$

With $\sigma = t - \tau$,

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma \quad .. (3)$$

Now, for the figure to the right:



$$x_1(t) = x(t).\, \delta_\Delta(t)$$

$$\text{As} \quad \Delta \to 0, \qquad x(t) = x(0)\text{, thus}$$

$$x_1(t) = x(t).\;\; \delta(t).$$

$$\text{As} \;\; \Delta \to 0,$$

$$x(t)\, \delta(t) = x(0) \;\; \delta(t). \qquad \text{... (4)}$$

In general:

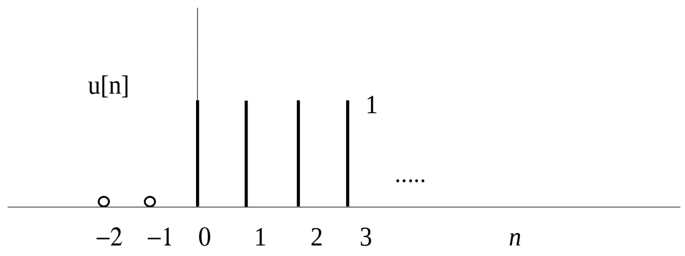
$$x(t)\, \delta(t - t_0) = x(t_0)\, \delta(t - t_0) \qquad \text{... (5)}$$

Discrete-Time:

Unit step function u[n]*:*

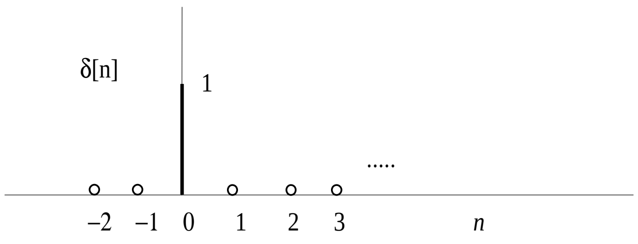
$$u[n] = 1 \qquad n \geq 0$$

$$= 0 \qquad n > 0$$



Unit Impulse (Unit Sample) function δ[n]:

$$\delta[n] = 1, \quad n = 0$$
$$= 0, \quad \text{otherwise.}$$



It is evident that:

$$u[n] - u[n-1] = \delta[n] \qquad \text{... (6)}$$

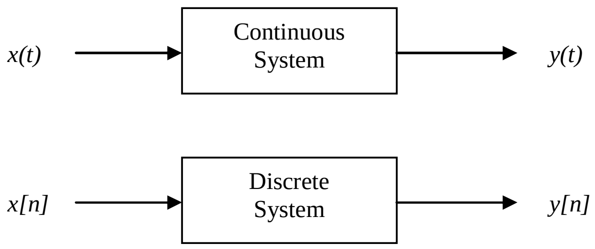
$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \qquad \text{... (7)}$$

Again, as in continuous-time,

$$x[n]\,\delta[n] = x[0]\,\delta[n] \qquad \text{... (8)}$$

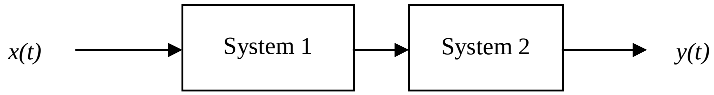
$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0] \qquad \text{... (9)}$$

Continuous- & Discrete Time Systems:

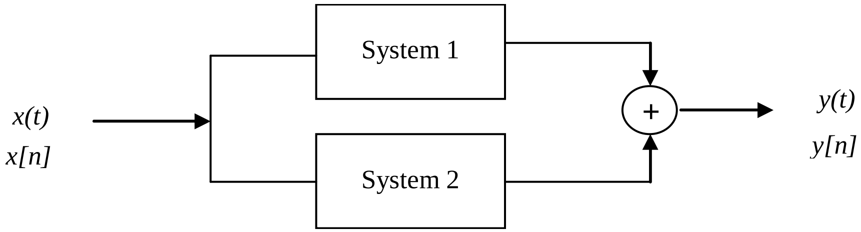


System Interconnections:

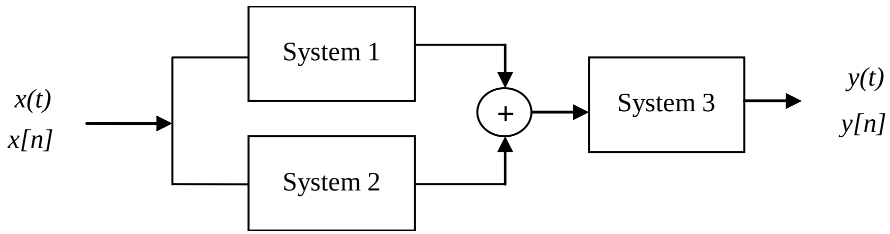
Cascaded/Series:



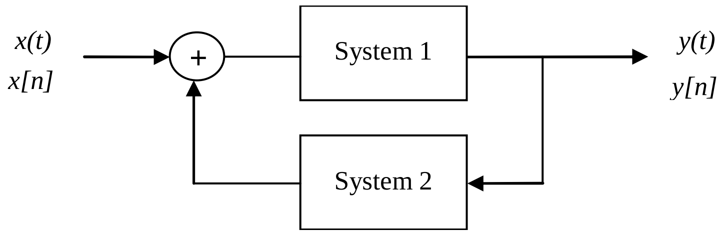
Parallel:



Series-Parallel:



Feedback:



BASIC System Classification:

1.Systems with and without Memory:

Memory-less:

- Output @ a time *t* depends on input @ the *same* time *t* only.

  Examples: y(t) = k. x(t) ;

  y[n] = k. x[n].

System with Memory:

- Output @ *t* depends on input @ *t* as well as @ *t < 0.*

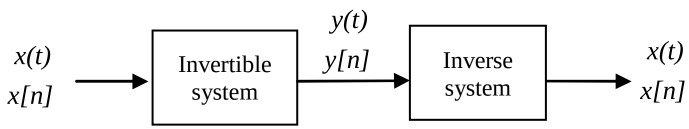
  Examples:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \; ; \qquad y[n] = \sum_{k=-\infty}^{n} x[k] \; ;$$

$$y[n] = y[n-2] + y[n-1] + x[n]$$

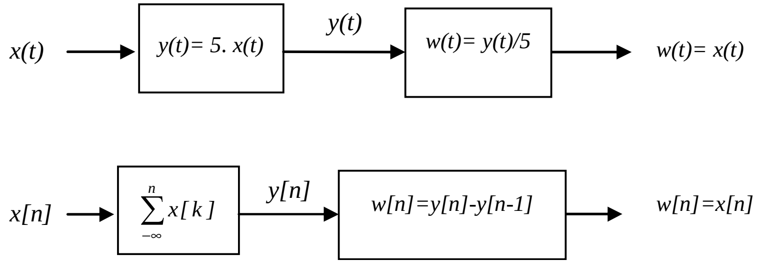
2. Invertibility and Inverse Systems:

- A system is invertible if *distinct* inputs produce *distinct* outputs.
- Produces original inputs when coupled with an *inverse* system.



Examples:

Inverting systems:



- *Non-inverting systems:*

  i) $y[n] = 0$. (A zero cannot reconstruct x[n])

  ii) $y(t) = x^2(t)$. (Sign of x(t) cannot be reconstructed).

Examples:

Prob. 1.30, p.62.

(a) $y(t) = x(t-4)$. Invertible. Inverse: $y(t) = x(t+4)$.

(b) $y(t) = \cos[x(t)]$. Non-invertible. (cos x(t) and cos $[x(t)+2k\pi]$) give the same output.

(d) $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$. Invertible. Inverse: $y(t) = \frac{d}{dt} x(t)$

(m) $y[n]= x[2n]$. Invertible. Inverse: $y[n]= x[n/2]$.

Causality:

• *Causal System*: output @ a time depends on the present and the past inputs only.

Important Properties of Causality:

• *If* $x(t) = 0$ *for* $t < t_0$ *, then*
$y(t) = 0$ *for* $t < t_0$.

Stability:

A system is stable if a *bounded* input produces a *bounded* output.

With an input $|x(t)| \le k_1$, if the output $|y(t)| \ge k_2$, k_1, k_2 are finite real, the system is stable.

Example of an unstable system:

$$y[n] = \sum_{n=-\infty}^{n} x[n]$$

; where $x[n] = u[n]$. ($y[n]$ increases without bound as n increases.)

Time Invariance:

•A system is time invariant if a time-shift in input produces the *same* time-shift in the output. For time invariance:

$$\text{If} \quad x(t) \to y(t), \quad \text{then} \quad x(t-t_0) \to y(t-t_0)$$

$$\text{If} \quad x[n] \to y[n], \quad \text{then} \quad x[n-n_0] \to y[n-n_0]$$

Examples:

Prob. 1.27 p.62:

(b) y(t) = [cos 3t].x(t).

R.H.S. = $x(t) \to x(t-t_0), \quad y(t-t_0) = [\cos 3t]\ x(t-t_0)$

L.H.S. = $y(t-t_0) = [\cos 3(t-t_0)]\ x(t-t_0)$

LHS $\neq$ RHs. Hence it is not time invariant.

(d)Not time invariant.

(e)Time invariant.

(f)Y(t) = x(t/3).

R.H.S.= $x(t-t_0) \to \quad y_1(t-t_0) = x(\frac{t}{3} - t_0)$

L.H.S. = $y_2(t-t_0) = x(\frac{(t-t_0)}{3}) = x(\frac{t}{3} - \frac{t_0}{3})$

$y_1 \neq y_2$. Not time invariant.

Prob. 1.28 p.62

(c) y[n] = n x[n].

R,H,S. : $y[n - n_0] \rightarrow nx[n - n_0]$

L.H.S. : $y[n - n_0] = (n - n_0)\, x[n - n_0]$

L.H.S. $\neq$ R.H.S. Not time invariant.

(b)Time invariant.

Linearity:

- A linear system obeys the principle of superposition.
- Satisfies the following properties:

➢ *Scaling:*

$$x(t) \rightarrow y(t)$$

$$a\, x(t) \rightarrow a\, y(t),$$

$$x[n] \rightarrow y[n]$$

$$a\, x[n] \rightarrow a\, y[n], \quad a \text{ is a complex constant.}$$

- *Additivity:*

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$

then, $a\, x_1(t) + b\, x_2(t) \to a\, y_1(t) + b\, y_2(t)$

$$a\, x_1[n] + b\, x_2[n] \to a\, y_1[n] + b\, y_2[n]$$

a and b are complex constants.

A few important Series:

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}, \quad (a \neq 1)$$

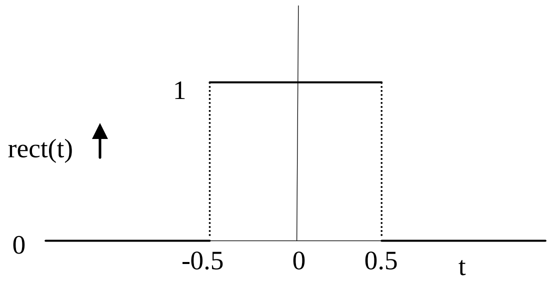
$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2}, \quad |a| < 1.$$

A few relevant functions:

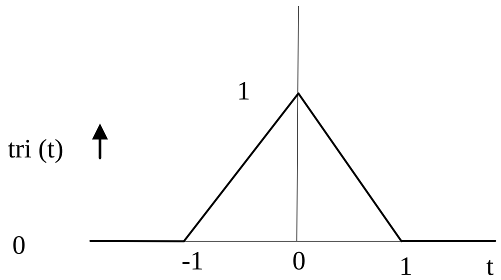
Rectangular Function rect(t):

$$\text{rect}(t) = \begin{cases} 0, & |t| > 0.5 \\ 0.5, & |t| = 0.5 \\ 1, & |t| < 0.5 \end{cases}$$



Triangular Function tri(t):

$$\text{tri}(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{else.} \end{cases}$$



Ramp Function R(t):

$$R(t) = \begin{cases} t, & t \geq 0 \\ 0 & t < 0. \end{cases}$$