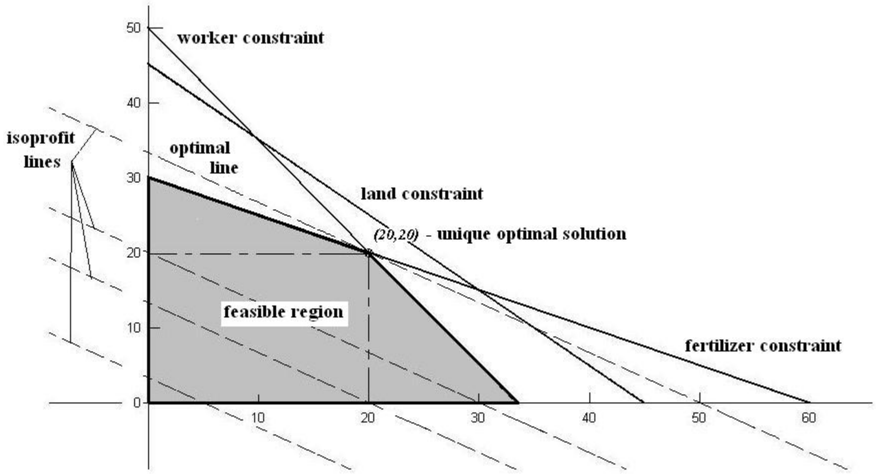
Assignment 8, Solutions
Geometry Approach and Analytic - Matrix Approach to Sensitivity Analysis

Problem 1. Farmer Jane has 45 acres of land. She is going to plant each acre of her land with wheat and corn. Each acre planted with wheat yields 200$ profit and needs 3 workers and 2 tons of fertilizer. Each acre planted with corn yields 300$ profit and needs 2 workers and 4 tons of fertilizer. One hundred (100) workers and 120 tons of fertilizer are available.
(1) Formulate an LPP whose solution tells farmer Jane how to maximize the profit from her land and solvethe LPP graphically. Determine the optimal basis graphically. Determine graphically, is the optimal solution of the LP unique? Explain by using your graphical solution.
(2) Solve the LPP obtained in (1) by using the Simplex Method. Determine the optimal basis by using the simplex solution. By using your simplex solution, is the optimal solution of the LpP unique - explain by using the simplex solution?
(3) By using the original LPP and knowing the optimal basis construct the optimal tableau of the Canoical Form of the LPP **without solving by the Simplex Method or graphically**. To this end use and appropriate matrix representation of the Canonical Form of the Initial (original) LPP obtained in (1).
(4) Instead of 200$ suppose that $c_1$$ is the contribution to profit from an acre of wheat. For what values of c_1 does the current basis remain optimal, i.e., the BV (basic variables) in the optimal solution will remain the same? Instead of 300$ suppose that c_2 is the contribution to profit from an acre of corn. For what values of c_2 does the current optimal basis remain optimal?
(5) Determine the Dual Prices (the Shadow Prices) of each constraint.
(6) Determine the range (the maximum increase and decrease) of the right hand side of each constraint such that the current optimal basis to remain optimal.

Solution. **(1)** Let x_1 denote the number of acres planted with wheat and x_2 denote the number of acres planted with corn. The the LP that is a mathematical model of farmer Jane situation is the following:

max $z = 200x_1 + 300x_2$	(objective function)
ST	
$3x_1 + 2x_2 \leq 100$	(workers constraint
$2x_1 + 4x_2 \leq 120$	(fertilizer constraint)
$x_1 + x_2 \leq 45$	(land constraint)
$x_1 \geq 0, x_2 \geq 0$	(sign restrictions)

Graphical solution and explanations on the graph below.



The feasible region of the problem is determined by $3x_1 + 2x_2 = 100$ $(w.c.)$; $2x_1 + 4x_2 = 120$ $(f.c.)$, $x_1 = 0$ $(sign\ restr.)$, $x_2 = 0$ $(sign\ restr.)$. Solving with respect to x_2 the isoprofit lines $z_0 = 200x_1 + 300x_2$, where z_0 takes any positive value), and $3x_1 + 2x_2 = 100$; $2x_1 + 4x_2 = 120$ we obtain

$$x_2 = -\frac{2}{3}x_1 + \frac{z_0}{300}; \quad x_2 = -\frac{3}{2}x_1 + 50(w.c); \quad x_2 = -\frac{1}{2}x_1 + 30(f.c)$$

we see that the slope of each isoprofit line is greater than the slope of $3x_1 + 2x_2 = 100$ and smaller than the slope of $2x_1 + 4x_2 = 120$. This means that the isoprofit line $z_0 = 200x_1 + 300x_2$ $(z_0 = 10000)$ passing through the point of intersection of $3x_1 + 2x_2 = 100$ and $2x_1 + 4x_2 = 120$ $(x_1 = 20, x_2 = 20)$ will entirely belong to the sector determined by these two lines and will have only one common point with the feasible region, namely the point of intersection of $3x_1 + 2x_2 = 100$ and $2x_1 + 4x_2 = 120$. Moreover, this will be the last isoprofit line to have a common intersection with the feasible region when we move the isoprofit lines in a direction of increasing of the profit z_0. In view of this we can conclude that the given LP has A UNIQUE optimal solution and to find this unique optimal solution simply we have to solve the linear system

$$\begin{aligned} 3x_1 + 2x_2 &= 100 \\ 2x_1 + 4x_2 &= 120 \end{aligned}$$

to obtain $x_1 = 20$, $x_2 = 20$ and from here $z_{max} = 200 \times 20 + 300 \times 20 = 10000$. This also means that the last isoprofit line before to leave the feasible region is $10000 = 200x_1 + 300x_2$. See the graph of the feasible region and the isoprofit lines.

Summing up, the unique optimal solution of the LP problem that is a mathematical model of a farmer Jane's situation is: $\mathbf{z_{max} = 10000}$, $\mathbf{x_1 = 20}$, and $\mathbf{x_2 = 20}$. The graphical solution also shows the basic variables (BV) in the optimal solution (**the Basis**), obviously it will be x_1, x_2, s_3 introducing the slack variables in the standard form (see below). Why? Because the graphical solution shows that the worker constraint and the fertilizer constraint will be binding (for them $s_1 = s_2 = 0$) and obviously the land constraint is not a binding one so, $s_3 > 0$.

(2) **Simplex solution.**
Canonical Form of the LPP obtained in (1):

$$\begin{aligned}&\max\ \ z = 200x_1 + 300x_2\\&\text{s.t.}\\&3x_1 + 2x_2 + s_1 = 100\\&2x_1 + 4x_2 + s_2 = 120\\&x_1 + x_2 + s_3 = 45\\&\overline{}\\&x_1, x_2, s_1, s_2, s_3 \geq 0\end{aligned}$$

Canonical Tableau 0:

z	x_1	$\mathbf{x_2}$	s_1	s_2	s_3	RHS	BV	r.t.
1	-200	-300	0	0	0	0	z=0	
0	3	2	1	0	0	100	$s_1 = 100$	50
0	2	4	0	1	0	120	$s_2 = 120$	30*
0	1	1	0	0	1	45	$s_3 = 45$	45

x_2 will enter the basis replacing s_2 in Row 2.
Canonical Tableau 1:

z	$\mathbf{x_1}$	x_2	s_1	s_2	s_3	RHS	BV	r.t.
1	-50	0	0	75	0	9000	z=9000	
0	2	0	1	-1/2	0	40	$s_1 = 40$	20*
0	1/2	1	0	1/4	0	30	$x_2 = 30$	60
0	1/2	0	0	-1/4	1	15	$s_3 = 15$	30

x_1 will enter the basis replacing s_1 in Row 1.
Canonical Tableau 2:

z	x_1	x_2	s_1	s_2	s_3	RHS	BV	r.t.
1	0	0	25	62.5	0	10000	z=10000	
0	1	0	1/2	-1/4	0	20	$x_1 = 20$	
0	0	1	-1/4	3/8	0	20	$x_2 = 20$	
0	0	0	-1/4	-1/8	1	5	$s_3 = 5$	

Canonical Tableau 2 is an Optimal Canonical Tableau giving the UNIQUE optimal b.f.s. (the coefficients of the NBV (s_1 and s_2) in Row 0 are different from 0):

$$\mathbf{z_{max} = 10000,\ BV : x_1 = 20, x_2 = 20, s_3 = 5,\ NBV : s_1 = s_2 = 0}$$

and in view of this the UNIQUE optimal solution of the given LP is:

$$z_{max} = 10000,\ x_1 = 20,\ x_2 = 20\,.$$

Hence, the optimal basis is (x_1, x_2, s_3).

(3) Representing the standard form of the LP in a matrix form separating the BV (x_1, x_2, s_3) we obtain:

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 120 \\ 45 \end{bmatrix}$$

The inverse matrix B^{-1} of the Basic matrix B:

$$B = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and B^{-1} is

$$B^{-1} = \begin{bmatrix} 1/2 & -1/4 & 0 \\ -1/4 & 3/8 & 0 \\ -1/4 & -1/8 & 1 \end{bmatrix}.$$

Now, multiplying on the left the above matrix equation by B^{-1} we obtain the Optimal Canonical Form up to Row 0:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 3/8 \\ -1/4 & -1/8 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 5 \end{bmatrix}$$

that is

$$\begin{array}{l} x_1 + 1/2s_1 - 1/4s_2 = 20 \\ x_2 - 1/4s_1 + 3/8s_2 = 20 \\ s_3 - 1/4s_1 - 1/8s_2 = 5 \end{array}.$$

In order to obtain Row 0 in the Optimal Canonical Tableau we substitute the BV in Row 0 as it follows:

$$z = 200x_1 + 300x_2 = 200(20 - 1/2s_1 + 1/4s_2) + 300(20 + 1/4s_1 - 3/8s_2) = 10000 - 25s_1 - 62.5s_2$$

and finally the Optimal Tableau is:

z	x_1	x_2	s_1	s_2	s_3	RHS	BV	r.t.
1	0	0	25	62.5	0	10000	z=10000	
0	1	0	1/2	-1/4	0	20	$x_1 = 20$	
0	0	1	-1/4	3/8	0	20	$x_2 = 20$	
0	0	0	-1/4	-1/8	1	5	$s_3 = 5$	

showing that we have a unique optimal b.f.s. for the LP in standard form that yields a unique optimal solution of the given LP:

$$z_{max} = 10000, \; x_1 = 20, \; x_2 = 20\,.$$

Solution 1 of (4) Geometry Approach. The feasible region of the LP is bounded by $3x_1+2x_2 = 100$ $(w.c)$; $2x_1 + 4x_2 = 120$ $(f.c)$, $x_1 = 0$ (*sign restr.*), $x_2 = 0$ (*sign restr*). Solving with respect to x_2 the isoprofit line $z_0 = c_1x_1 + 300x_2(z_0 = 20c_1 + 6000)$ that is passing through the optimal solution ($x_1 = 20, x_2 = 20$), and solving also $3x_1 + 2x_2 = 100$; $2x_1 + 4x_2 = 120$ with respect to x_2 we obtain the slopes of the lines:

$$x_2 = -\frac{c_1}{300}x_1 + \frac{z_0}{300}; \quad x_2 = -\frac{3}{2}x_1 + 50; \quad x_2 = -\frac{1}{2}x_1 + 30.$$

Then, in order the current basis to remain optimal we must have that the isoprofit lines $z_0 = c_1x_1 + 300x_2$ will reveal the optimal solution ($x_1 = 20, x_2 = 20$) with the same optimal basis x_1, x_2, s_3 and this holds iff the isoprofit line $z = c_1x_1 + 300x_2$ passing through ($x_1 = 20, x_2 = 20$) remains in the sector determined by the two binding constraints $3x_1 + 2x_2 = 100$; $2x_1 + 4x_2 = 120$, i.e., iff

$$-\frac{3}{2} \leq -\frac{c_1}{300} \leq -\frac{1}{2}\,.$$

The above inequalities give the range of c_1: $\mathbf{450 \geq c_1 \geq 150}$. In other words **the allowable increase of the profit from wheat is 250\$ and the allowable decrease of the profit from wheat is 50\$. This means that the farmer Jane can increase the prise for sale of wheat and to achieve 450\$ of profit per acre planted with wheat remaining the same (optimal conditions, land planted) of farming.** Of course farmer Jane might be limited following the above optimal results from some other restrictions of the farming, market considerations, Government restrictions, etc., that are not taken into account in our simple LP.

Solving with respect to x_2 the isoprofit line $z_0 = 200x_1 + c_2x_2$ that is passing through the optimal solution ($x_1 = 20, x_2 = 20$), and solving $3x_1 + 2x_2 = 100$; $2x_1 + 4x_2 = 120$ with respect to x_2 we obtain

$$x_2 = -\frac{200}{c_2}x_1 + \frac{z_0}{c_2}; \quad x_2 = -\frac{3}{2}x_1 + 50; \quad x_2 = -\frac{1}{2}x_1 + 30.$$

Then, in order the current basis to remain optimal we must have that the isoprofit line $z_0 = 200x_1 + c_2x_2$ will reveal the same optimal solution ($x_1 = 20, x_2 = 20$) and this holds iff

$$-\frac{3}{2} \leq -\frac{200}{c_2} \leq -\frac{1}{2}$$

which gives the range of c_2: $\mathbf{400 \geq c_2 \geq 133.33}$ in other words **the allowable increase of the profit from corn is 100\$ and the allowable decrease of the profit from corn is 166.67\$. This means that the farmer Jane can increase the prise for sale of corn and to achieve 400\$ of profit for an acre planted with corn remaining the same acres of land planted that gives an optimal framing.**

Solution 2 of (4). Analytic-Matrix Approach. Assume that c_1\$ is the contribution to profit from an acre of wheat and 300 \$ is the profit from an acre of corn. Consider the modified goal function $z = c_1x_1 + 300x_2$. Because we do not change anything in constraints the Optimal Row 1-Row 3 will be

$$\begin{array}{l} x_1 + 1/2s_1 - 1/4s_2 = 20 \\ x_2 - 1/4s_1 + 3/8s_2 = 20 \\ s_3 - 1/4s_1 - 1/8s_2 = 5 \end{array} .$$

and eliminating from the goal function the BV we obtain Row 0 of the Optimal Tableau in the modified farmer Jane LP:

$$z = c_1(20-1/2s_1+1/4s_2)+300(20+1/4s_1-3/8s_2) = [6000+20c_1]+(75-c_1/2)s_1+(c_1/4-900/8)s_2\,.$$

Now, in order the above Row 0 to be Optimal we must have

$$75 - c_1/2 \le 0 \qquad c_1/4 - 450/4 \le 0 \Rightarrow \mathbf{150 \le c_1 \le 450}$$

and from here the range of c_1 is: $\mathbf{150 \le c_1 \le 450}$ in order the current basis (x_1, x_2, s_3) to remain optimal. Also, $\mathbf{z_{max}} = [\mathbf{6000 + 20c_1}]$\$ for each contribution to profit c_1\$ in the interval $[150, 450]$.
Similarly, assume that c_2\$ is the contribution to profit from an acre of corn and 200 \$ is the profit from an acre of wheat. Consider the modified goal function $z = 200x_1 + c_2x_2$. We do not change anything in the constraints hence, the Optimal Row 1-Row 3 will be

$$\begin{array}{l} x_1 + 1/2s_1 - 1/4s_2 = 20 \\ x_2 - 1/4s_1 + 3/8s_2 = 20 \\ s_3 - 1/4s_1 - 1/8s_2 = 5 \end{array} .$$

and eliminating from the goal function the BV we obtain Row 0 of the Optimal Tableau in the modified farmer Jane LP:

$$z = 200(20 - 1/2s_1 + 1/4s_2) + c_2(20 + 1/4s_1 - 3/8s_2) = [4000 + 20c_2] + (c_2/4 - 100)s_1 + (50 - 3/8c_2)s_2 .$$

Now, in order the above Row 0 to be Optimal we must have

$$c_2/4 - 100 \le 0 \qquad 50 - 3/8c_2 \le 0 \Rightarrow \mathbf{133.3 \le c_2 \le 400}$$

and from here the range of c_2 is: $\mathbf{133.3 \le c_2 \le 400}$ in order the current basis (x_1, x_2, s_3) to remain optimal. Also, $\mathbf{z_{max}} = [\mathbf{4000 + 20c_2}]$\$ for each contribution to profit c_2\$ from acre of corn in the interval $[133.3, 400]$.

Solution 1 of (5). Geometry Approach. First the shadow prices are different from zero only for the binding constraints. In our LP the binding constraints are $3x_1 + 2x_2 = 100$ (workers constraint) and $2x_1 + 4x_2 = 120$ (fertilizer constraint).
To determine **the shadow price of the workers constraint** we solve:

$$\begin{array}{l} 3x_1 + 2x_2 = 100 + \Delta \\ 2x_1 + 4x_2 = 120 \end{array}$$

We have $x_1 = 20 + \Delta/2$, $x_2 = 20 - \Delta/4$ and

$$z = 200(20 + \Delta/2) + 300(20 - \Delta/4) = 10000 + 100\Delta - 75\Delta = 10000 + 25\Delta.$$

Hence, increasing (decreasing) the workers with Δ will increase (decrease) the profit with 25Δ \$ Dollars. **From here, the shadow price for the workers binding constraint is 25\$, the coefficient in front of Δ in the goal function above. Of course the increasing or the decreasing of the righthand side must be in the allowable increase or decrease such that the optimal basis (x_1, x_2, s_3) will remain unchanged (the solution of (6) below gives that in order the current basis to remain optimal we must have $-40 \le \Delta \le 20$).**

To determine **the shadow price of the fertilizer constraint** we solve:

$$\begin{array}{l} 3x_1 + 2x_2 = 100 \\ 2x_1 + 4x_2 = 120 + \Delta \end{array}$$

We have $x_1 = 20 - \Delta/4$ $x_2 = 20 + 3\Delta/8$ and

$$z = 200(20 - \Delta/4) + 300(20 + 3\Delta/8) = 10000 - 50\Delta + 112.5\Delta = 10000 + 62.5\Delta.$$

Hence, increasing (decreasing) the fertilizer with Δ tons will increase (decrease) the profit with 62.5Δ\$. **The shadow price for the fertilizer (binding) constraint is 62.5\$. Of course the increasing or the decreasing of the righthand side must be in the allowable increase or decrease such that the optimal basis x_1, x_2, s_3 will remain unchanged (see the solution of (6) below to see that in order the current basis to remain optimal we must have $-53.3 \leq \Delta \leq 40$).**

Third, the shadow price of the land constraint is 0, because it is non-binding constraint. In its allowable increase or decrease the optimal values of x_1 and x_2 remain unchanged because the binding constraint remain unchanged.

Solution 1 of (6). Geometry Approach.
First, consider **the worker's constraint** $3x_1 + 2x_2 \leq 100$. Graphically (see the graphs of the feasible region and the constraints) it is clear that the current basis (x_1, x_2, s_3) will remain optimal for $3x_1 + 2x_2 \leq b_1$ when the line $3x_1 + 2x_2 = b_1$ (1) is passing through $(x_1 = 0, x_2 = 30)$ and for $3x_1 + 2x_2 \leq b_1$ when the line $3x_1 + 2x_2 = b_1$ (2) is passing through the point of intersection $(x_1 = 30, x_2 = 15)$ of $2x_1 + 4x_2 = 120$ and $x_1 + x_2 = 45$ and in addition, for all workers restriction $3x_1 + 2x_2 \leq b_1$ with boundary lines $3x_1 + 2x_2 = b_1$ that in sector determined by the lines (1) and (2). Hence, $3x_1 + 2x_2 \leq 60$ and $3x_1 + 2x_2 \leq 120$ are the limiting working constraint lines in order the current basis (x_1, x_2, s_3) to remain optimal. **In view of this the maximum increase of the righthand side of worker's constraint is 20 workers and the maximum decrease is 40 workers. For bigger increase $>$ 20 and smaller decrease $<$ 40 the optimal basis will change, i.e., the current bases will be no longer optimal.**

Second, consider **the fertilizer's constraint** $2x_1 + 4x_2 \leq 120$. Graphically (see the graphs of the feasible region and the constraints) it is clear that the current basis (x_1, x_2, s_3) will remain optimal for $2x_1 + 4x_2 \leq b_2$ with boundary line $2x_1 + 4x_2 = b_2$ passing through $(x_1 = 100/3, x_2 = 0)$ and for $2x_1 + 4x_2 \leq b_2$ with boundary line $2x_1 + 4x_2 = b_2$ passing through the point of intersection $x_1 = 10, x_2 = 35$ of $3x_1 + 2x_2 = 100$ and $x_1 + x_2 = 45$ and for all fertilizer constraints $2x_1 + 4x_2 \leq b2$ with boundary lines $2x_1 + 4x_2 = b2$ that are between these two. Hence, $2x_1 + 4x_2 = 66.7$ and $2x_1 + 4x_2 = 160$ are the limiting fertilizer constraint lines in order the current basis (x_1, x_2, s_3) to remain optimal. **Hence the maximum increase of the righthand side of fertilizer constraint is 40 tons and the maximum decrease is 53.3 tons. For bigger than 40 tons increase and smaller than 53.3 tons decrease of the righthand side of the fertilizer constraints the optimal basis will change, i.e., the current basis (x_1, x_2, s_3) will be no longer optimal.**

The land constraint. Obviously, see the graph of the feasible region and the land constraint, the current basis (x_1, x_2, s_3) will remain optimal if the total acres of planted land is ≥ 40. If $x_1 + x_2 < 40$ then the basis will change. On the other hand if $x_1 + x_2 \geq 40$ then the current optimal basis (x_1, x_2, s_3) will remain unchanged even the unique optimal solution $(x_1 = 20, x_2 = 20)$ will stay the same. **Hence, the allowable decrease of the righthand side of the land constraint is 5 acres and the allowable increase of the righthand side of the land constraint is ∞ acres.**

Solution 2 of (5) and (6) by using Analytic-Matrix Approach. By using Analytic-Matrix Approach to Sensitivity Analysis we shall solve (5) and (6) simultaneously for each of

the constraints. Determine the Shadow (Dual) Price; determine the maximal (Allowable) increase, and the maximal (Allowable) Decrease of each constraint.
Solution. Worker Constraint. Suppose that the righthand side of the worker's constraint has been changed to $100 + \Delta$. Then, the change of the righthand side in the Optimal Tableau will be

$$\begin{bmatrix} 1/2 & -1/4 & 0 \\ -1/4 & 3/8 & 0 \\ -1/4 & -1/8 & 1 \end{bmatrix} \begin{bmatrix} 100 + \Delta \\ 120 \\ 45 \end{bmatrix} = \begin{bmatrix} 20 + \Delta/2 \\ 20 - \Delta/4 \\ 5 - \Delta/4 \end{bmatrix}$$

and for the Row 0 of the modified LP we obtain

$$z = 200(20 + \Delta/2 - 1/2s_1 + 1/4s_2) + 300(20 - \Delta/4 + 1/4s_1 - 3/8s_2) = 10000 + \mathbf{25}\Delta - 25s_1 - 62.5s_2$$

and from here the Row 0 is

$$\mathbf{z + 25s_1 + 62.5s_2 = 10000 + 25\Delta}\,.$$

Now applying the trivial fact that all righthand sides in any Tableau including the Optimal Tableau must be non-negative we obtain

$$20 + \Delta/2 \geq 0; \quad 20 - \Delta/4 \geq 0; \quad 5 - \Delta/4 \geq 0 \quad \Rightarrow \quad \mathbf{-40 \leq \Delta \leq 20}$$

and from here **the maximal increase is 20 workers and the maximal decrease is 40 workers in order the current basis to remain optimal. The coefficient 25 of $\boldsymbol{\Delta}$ in the Row 0 of the Optimal Tableau of the Modified LP with worker's righthand side** $(100 + \Delta)$ **reveals the shadow price of the worker's constraint: 25\$.**

Fertilizer Constraint. Suppose that the righthand side of the fertilizer's constraint has been changed to $120 + \Delta$. Then, the change of the righthand side in the Optimal Tableau will be

$$\begin{bmatrix} 1/2 & -1/4 & 0 \\ -1/4 & 3/8 & 0 \\ -1/4 & -1/8 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 120 + \Delta \\ 45 \end{bmatrix} = \begin{bmatrix} 20 - \Delta/4 \\ 20 + 3\Delta/8 \\ 5 - \Delta/8 \end{bmatrix}$$

and for the Row 0 of the modified LP we obtain

$$z = 200(20 - \Delta/4 - 1/2s_1 + 1/4s_2) + 300(20 + 3\Delta/8 + 1/4s_1 - 3/8s_2) = 10000 + \mathbf{65.5}\Delta - 25s_1 - 62.5s_2$$

and from here the Row 0 is

$$\mathbf{z + 25s_1 + 62.5s_2 = 10000 + 65.5\Delta}\,.$$

Now applying the trivial fact that all righthand sides in any Tableau including the Optimal Tableau must be non-negative we obtain

$$20 - \Delta/4 \geq 0; \quad 20 + 3\Delta/8 \geq 0; \quad 5 - \Delta/8 \geq 0 \quad \Rightarrow \quad \mathbf{-53.3 \leq \Delta \leq 40}$$

and from here **the maximal increase is 40 tons of fertilizer and the maximal decrease is 53.3 tons of fertilizer in order the current basis to remain optimal. The coefficient 65.5 of $\boldsymbol{\Delta}$ in the Row 0 of the Optimal Tableau of the Modified LP with fertilizer's righthand side** $(120 + \Delta)$ **reveals the shadow price of the fertilizer's constraint: 65.5\$.**

Land constraint. Suppose that the righthand side of the land's constraint has been changed to $45+\Delta$. Then, the change of the righthand side in the Optimal Tableau will be

$$\begin{bmatrix} 1/2 & \text{-}1/4 & 0 \\ \text{-}1/4 & 3/8 & 0 \\ \text{-}1/4 & \text{-}1/8 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 120 \\ 45+\Delta \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 5+\Delta \end{bmatrix}$$

and for the Row 0 of the modified LP we obtain

$$z = 200(20 - 1/2s_1 + 1/4s_2) + 300(20 + 1/4s_1 - 3/8s_2) = 10000 + -25s_1 - 62.5s_2$$

and from here the Row 0 is

$$\mathbf{z + 25s_1 + 62.5s_2 = 10000 + 0\Delta}\,.$$

Now applying the trivial fact that all righthand sides in any Tableau including the Optimal Tableau must be non-negative we obtain

$$20 \geq 0; \quad 20 \geq 0; \quad 5+\Delta \geq 0 \quad \Rightarrow \quad \mathbf{-5 \leq \Delta} < \infty$$

and from here **the maximal increase is ∞ acres of land and the maximal decrease is 5 acres of land in order the current basis to remain optimal. The coefficient 0 of Δ in the Row 0 of the Optimal Tableau of the Modified LP with land's righthand side** $(45+\Delta)$ **reveals the shadow price of the land's constraint: 0$.**
We can conclude without any computations that the shadow price of the land constraint is 0$ **because the land constraint is not a binding one.**

Problem 2. Bloomington Breweries produces beer and ale. Beer sells for 3$ a barrel and ale sells for 4$ a barrel. Producing a barrel of beer needs 3 lb of barley and 2 lb of hops. Producing a barrel of ale needs 2 lb of barley and 3 lb of hops. Marketing constraint requires at most a total of 50 barrels to be produced weekly. Hundred (100) lb of barley and 100 lb of hops are available each week.
(a) Formulate an LPP whose solution tells Bloomington Breweries how to maximize the profit of its production.
(b) Solve the LPP from **(a)** by using the Extreme Point Theorem and find the optimal basis.
(c) Instead of 3$ suppose that a barrel of beer sells for $c_1$$. For what values of c_1 does the current optimal basis remain optimal?
Instead of 4$ suppose that a barrel of ale sells for $c_2$$. For what values of c_2 does the current optimal basis remain optimal?
(d) Find the shadow (dual) prices of each constraint by using matrix approach to sensitivity analysis.
(e) Use matrix approach to sensitivity analysis to determine the maximal (allowable) increase and the maximal (allowable) decrease of the right hand side of each constraint such that the current optimal basis will remain optimal.
Solution of (a). Decision variables: The number x_1 of barrels of beer produced each week and the number x_2 of barrels of ale produced each week. Then, the LPP that is a Mathematical Model of Bloomington Breweries situation in terms of the decision variables x_1, x_2 is the following:

max $z = 3x_1 + 4x_2$	(objective function)
s.t.	
$3x_1 + 2x_2 \leq 100$	(barley constraint
$2x_1 + 3x_2 \leq 100$	(hops constraint)
$x_1 + x_2 \leq 50$	(marketing constraint)
$x_1 \geq 0, x_2 \geq 0$	(sign restrictions)

(b) Solve the LPP from **(a)** by using the Extreme Point Theorem. Find the optimal basis.
Solution of (b). The feasible region is bounded hence, an optimal solution exists. By the Extreme Point Theorem we must have an optimal corner (extreme) point solution. The corner points are $(0,0), (100/3, 0), (20, 20), (0, 100/3)$ and the values of the objective function at the corner points are $z(0,0) = 0, z(100/3, 0) = 100, z(20, 20) = 140, z(0, 100/3) = 400/3$ hence,

$$z_{max} = 140,\ x_1 = 20,\ x_2 = 20$$

is the unique optimal solution. Why? Note that, according to the divisibility condition for an LPP, the optimal values of x_1 and x_2 must be integer numbers, we can not produce a half or a quoter of barrel of beer (ale). By using the Canonical Form of the LPP

$$\begin{array}{ll}
\max \ \ z = 3x_1 + 4x_2 & \text{(objective function)} \\
\text{ST} & \\
3x_1 + 2x_2 + s_1 = 100 & \text{(barley constraint} \\
2x_1 + 3x_2 + s_2 = 100 & \text{(hops constraint)} \\
x_1 + x_2 + s_3 = 50 & \text{(marketing constraint)} \\
\hline
x_1 \geq 0, x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0 & \text{(sign restrictions)}
\end{array}$$

we compute the optimal BFS: BV $x_1 = 20,\ x_2 = 20, s_3 = 10$; NBV $s_1 = s_2 = 0$ Hence, the optimal basis is (x_1, x_2, s_3).
In order to solve (c), (d), and (e) we construct the optimal canonical tableau of the LPP by using the initial canonical matrix form of the LPP and the fact that (x_1, x_2, s_3) **is the optimal basis.** Representing the initial canonical form of the LP in a matrix form separating the optimal basis (x_1, x_2, s_3) we obtain:

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 50 \end{bmatrix}$$

The inverse matrix B^{-1} of the Basic matrix B:

$$B = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 3/5 & -2/5 & 0 \\ -2/5 & 3/5 & 0 \\ -1/5 & -1/5 & 1 \end{bmatrix}.$$

Multiplying on the left the above matrix equation by B^{-1} we obtain the Optimal Canonical Tableau up to the Row 0:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} 3/5 & -2/5 \\ -2/5 & 3/5 \\ -1/5 & -1/5 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 10 \end{bmatrix}$$

that is

$$\begin{array}{ll}
\text{Row 1} & x_1 + 3/5 s_1 - 2/5 s_2 = 20 \\
\text{Row 2} & x_2 - 2/5 s_1 + 3/5 s_2 = 20 \\
\text{Row 3} & s_3 - 1/5 s_1 - 1/5 s_2 = 10
\end{array}.$$

In order to obtain Row 0 in the Optimal Canonical Tableau we substitute the BV in Row 0 as it follows:

$$z = 3x_1 + 4x_2 = 3(20 - 3s_1/5_1 + 2s_2/5) + 4(20 + 2s_1/5 - 3s_2/5) = 140 - \frac{1}{5}s_1 - \frac{6}{5}s_2$$

and from here the Row 0 in the optimal tableau is:

$$z + \frac{1}{5}s_1 + \frac{6}{5}s_2 = 140\,.$$

and finally the Optimal Canonical Tableau is:

	z	x_1	x_2	s_1	s_2	s_3	RHS	BV
Row 0	1	0	0	1/5	6/5	0	140	z=140
Row 1	0	1	0	3/5	-2/5	0	20	$x_1 = 20$
Row 2	0	0	1	-2/5	3/5	0	20	$x_2 = 20$
Row 3	0	0	0	-1/5	-1/5	1	10	$s_3 = 10$

showing that we have a unique (**the coefficients of the NBV in Row 0 are strictly positive**) optimal BFS: BV $z_{max} = 140, x_1 = 20, x_2 = 20, s_3 = 10$; NBV $s_1 = s_2 = 0$ for the LPP in canonical form that yields a unique optimal solution of the given LPP: $z_{max} = 140,\ x_1 = 20,\ x_2 = 20\,.$

(c) Instead of 3$ suppose that a barrel of beer sells for c_1\$. For what values of c_1 does the current basis remain optimal? Instead of 4$ suppose that a barrel of ale sells for c_2\$. For what values of c_2 does the current basis remain optimal?

Solution of (c). Assume that c_1\$ is the contribution to profit from a barrel of beer and 4 $ is the profit from a barrel of ale. Consider the modified objective function $z = c_1x_1 + 4x_2$. Because we do not change anything in the constraints the Optimal Row 1-Row 3 will be

$$\begin{array}{ll} \text{Row 1} & x_1 + 3/5s_1 - 2/5s_2 = 20 \\ \text{Row 2} & x_2 - 2/5s_1 + 3/5s_2 = 20 \\ \text{Row 3} & s_3 - 1/5s_1 - 1/5s_2 = 10 \end{array}\,.$$

and eliminating from the goal function the BV we obtain Row 0 of the optimal tableau in the Modified Bloomington Breweries LPP:

$$z = c_1(20 - \frac{3}{5}s_1 + \frac{2}{5}s_2) + 4(20 + \frac{2}{5}s_1 - \frac{3}{5}s_2) = [80 + 20c_1] + (\frac{8}{5} - \frac{3}{5}c_1)s_1 + (\frac{2}{5}c_1 - \frac{12}{5})s_2\,.$$

Now, in order the above Row 0 to be Optimal we must have

$$\frac{8}{5} - \frac{3}{5}c_1 \le 0 \qquad \frac{2}{5}c_1 - \frac{12}{5} \le 0 \Rightarrow \mathbf{\frac{8}{3} \le c_1 \le 6}$$

and from here: $\mathbf{2.67}\$ \le \mathbf{c_1} \le \mathbf{6}\$$ in order the current basis (x_1, x_2, s_3) to remain optimal. Also, $\mathbf{z_{max}} = [\mathbf{80 + 2c_1}]\$$ for a contribution to profit c_1\$ in the interval $[2.67, 6]$ from a barrel of beer and 4$ for a barrel of ale.
Similarly, assume that c_2\$ is the contribution to profit from a barrel of ale and 3 $ is the profit from a barrel of beer. Consider the modified goal function $z = 3x_1 + c_2x_2$. Because we do not change anything in constraints the Optimal Row 1-Row 3 will be

$$\begin{array}{ll} \text{Row 1} & x_1 + 3/5s_1 - 2/5s_2 = 20 \\ \text{Row 2} & x_2 - 2/5s_1 + 3/5s_2 = 20 \\ \text{Row 3} & s_3 - 1/5s_1 - 1/5s_2 = 10 \end{array}\,.$$

and eliminating from the goal function the BV we obtain Row 0 of the optimal tableau in the Modified Bloomington Breweries LPP:

$$z = 3(20 - \frac{3}{5}s_1 + \frac{2}{5}s_2) + c_2(20 + \frac{2}{5}s_1 - \frac{3}{5}s_2) = [60 + 20c_2] + (\frac{2}{5}c_2 - \frac{9}{5})s_1 + (\frac{6}{5} - \frac{3}{5}c_2)s_2\,.$$

Now, in order the above Row 0 to be Optimal we must have

$$\frac{2}{5}c_2 - \frac{9}{5} \le 0 \qquad \frac{6}{5} - \frac{3}{5}c_2 \le 0 \Rightarrow \mathbf{2 \le c_2 \le \frac{9}{2}}$$

and from here the range of c_2 is $\mathbf{2}\$ \le \mathbf{c_2} \le \mathbf{4.5}\$$ in order the current basis (x_1, x_2, s_3) to remain optimal. Also, $\mathbf{z_{max}} = [\mathbf{60 + 20c_2}]\$$ for each contribution to profit $c_2\$$ in the interval $[2, 4.5]$ of a barrel of ale and a contribution to profit 3$ of a barrel of beer.

(d) Find the shadow (dual) prices of each constraint by using matrix approach to sensitivity analysis.

(e) Use matrix approach to sensitivity analysis to determine the maximal (allowable) increase and the maximal (allowable) decrease of the right hand side of each constraint such that the current optimal basis to remain optimal.

Solution of (d) and (e) Applying matrix approach to sensitivity analysis we solve (d) and (e) simultaneously for each of the constraints. Determine the shadow (dual) price, the maximal (allowable) increase, and the maximal (allowable) decrease of each constraint. In our solution the crucial role will play the inverse B^{-1} of the basic matrix B.
Solution.
Barley constraint. Suppose that the righthand side of the barley's constraint has been changed to $100 + \Delta$. Then, the change of the righthand side in the Optimal Tableau will be

$$\begin{bmatrix} 3/5 & -2/5 & 0 \\ -2/5 & 3/5 & 0 \\ -1/5 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} 100+\Delta \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 20 + 3\Delta/5 \\ 20 - 2\Delta/5 \\ 10 - \Delta/5 \end{bmatrix}$$

and for the Row 0 of the Modified LPP we obtain

$$z = 3(20 + 3\Delta/5 - \frac{3}{5}s_1 + \frac{2}{5}s_2) + 4(20 - 2\Delta/5 + \frac{2}{5}s_1 - \frac{3}{5}s_2) = 140 + \frac{\mathbf{1}}{\mathbf{5}}\Delta - \frac{1}{5}s_1 - \frac{6}{5}s_2$$

and from here the Row 0 is

$$\mathbf{z + 0.2\,s_1 + 1.2\,s_2 = 140 + 0.2\Delta}\,.$$

Now applying the trivial fact that all righthand sides in any Tableau (including the Optimal Tableau) must be non-negative we obtain

$$20 + 3\Delta/5 \ge 0; \quad 20 - 2\Delta/5 \ge 0; \quad 10 - \Delta/5 \ge 0 \quad \Rightarrow \quad \mathbf{-100/3 \le \Delta \le 50}$$

and from here **the maximal increase is 50 lb of barley and the maximal decrease is 33.3 lb of barley in order the current basis to remain optimal. The coefficient 0.2 in front of Δ in the Row 0 of the Optimal Tableau of the Modified LPP with barley's righthand side $(100 + \Delta)$ reveals the shadow price of the barley constraint: 0.2$.**

Hops constraint. Suppose that the righthand side of the hops constraint has been changed to $100 + \Delta$. Then, the change of the righthand side in the optimal tableau will be

$$\begin{bmatrix} 3/5 & -2/5 & 0 \\ -2/5 & 3/5 & 0 \\ -1/5 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 100+\Delta \\ 50 \end{bmatrix} = \begin{bmatrix} 20 - 2\Delta/5 \\ 20 + 3\Delta/5 \\ 10 - \Delta/5 \end{bmatrix}$$

and for the Row 0 of the modified LP we obtain

$$z = 3(20 - 2\Delta/5 - \frac{3}{5}s_1 + \frac{2}{5}s_2) + 4(20 + 3\Delta/5 + \frac{2}{5}s_1 - \frac{3}{5}s_2) = 140 + \mathbf{\frac{6}{5}}\Delta - \frac{1}{5}s_1 - \frac{6}{5}s_2$$

and from here the Row 0 is

$$\mathbf{z + 0.2\, s_1 + 1.2\, s_2 = 140 + 1.2\Delta}\,.$$

Now applying the trivial fact that all righthand sides in any Tableau (including the Optimal Tableau) must be non-negative we obtain

$$20 - 2\Delta/5 \geq 0; \quad 20 + 3\Delta/5 \geq 0; \quad 10 - \Delta/5 \geq 0 \quad \Rightarrow \quad \mathbf{-100/3 \leq \Delta \leq 50}$$

and from here **the maximal increase is 50 lb of hops and the maximal decrease is 33.3 lb of hops in order the current basis to remain optimal. The coefficient 1.2 in front of Δ in the Row 0 of the Optimal Tableau of the Modified LP with hops righthand side** $(100 + \Delta)$ **reveals the shadow price of the hops constraint: 1.2$.**

Marketing constraint. Suppose that the righthand side of the marketing constraint has been changed to $50 + \Delta$. Then, the change of the righthand side in the optimal tableau will be

$$\begin{bmatrix} 3/5 & \text{-}2/5 & 0 \\ \text{-}2/5 & 3/5 & 0 \\ \text{-}1/5 & \text{-}1/5 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \\ 50 + \Delta \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 10 + \Delta \end{bmatrix}$$

and for the Row 0 of the Modified LPP we obtain

$$z = 3(20 - \frac{3}{5}s_1 + \frac{2}{5}s_2) + 4(20 + \frac{2}{5}s_1 - \frac{3}{5}s_2) = 140 - \frac{1}{5}s_1 - \frac{6}{5}s_2$$

and from here the Row 0 is

$$\mathbf{z + 0.2\, s_1 + 1.2\, s_2 = 140 + 0\Delta}\,.$$

Now applying the trivial fact that all righthand sides in any Tableau (including the Optimal Tableau) must be non-negative we must have

$$20 \geq 0; \quad 20 \geq 0; \quad 10 + \Delta \geq 0 \quad \Rightarrow \quad \mathbf{-10 \leq \Delta \leq \infty}$$

and from here **the maximum increase is ∞ barrels and the maximum decrease is 10 barrels in total in order the current basis to remain optimal. The coefficient 0 in front of Δ in the Row 0 of the Optimal Tableau of the Modified LP with barrels right hand side** $(50 + \Delta)$ **reveals the shadow price of the marketing (the total number of barrels) constraint: 0$.**

The fact that the shadow price of the marketing constraint is 0$ is trivial, taking into account that the marketing constraint is not binding for the optimal solution.