

Linear Algebra Method to Solve LPP by using the Extreme Point Theorem

Canonical Form of a Linear Programming Problem.

Basic Solutions (BS) and Basic Feasible Solutions (BFS) of a Canonical LPP. Basic and Non-basic Variables of a BFS.

Algebraic Computing of the Extreme (Corner) Points of a Feasible Region.

In the previous lectures we discussed the geometry of the feasible regions of two-variable and three-variable LPPs. We have used geometry to find the extreme (corner) points of a feasible region of a given LPP and we have applied geometry (graphical) methods to solve a given LPP. We have seen the importance of finding the extreme (corner) points of the feasible region of a given LPP in order to solve a given LPP.

However, it is not possible to draw (to visualize) the feasible region of a four- or more-variable LPP. In view of this it is impossible to visualize the extreme (corner) points of the feasible region of an LPP having many variables. Even for three-variable LPPs to visualize the feasible region and its extreme (corner) points and to parallel move iso-profit (iso-cost) planes in order to find an optimal feasible solution by using geometry is a difficult task that can not be modeled in a simple algorithm.

In this section we develop Linear Algebra Method to compute the extreme (corner) points of the FR of a given LPP. This Linear Algebra Method is applicable for LPPs with many variables and many constraints (in fact arbitrary number of decision variables and constraints).

The Linear Algebra Method is based on the notions of a Basic Solution and a Basic Feasible Solution of a Canonical LPP and permits a simple algorithmic modeling.

Proposition 1. If in a given n -variable, m -constraint LPP all variables are non-negatively sign restricted i.e., $x_i \geq 0, i = 1, 2, \dots, n$, then the given LPP and its Standard Form (the corresponding standard LPP) coincide: Both, the given LPP and its Standard Form have the same feasible region (FR), the same set of extreme corner points, the same objective function, and the same set of optimal solutions.

Proof. Suppose that the given LPP has a constraint in the form:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq (\geq) (=) b_i, \quad b_i < 0.$$

Multiplying by -1 the above inequality we obtain an equivalent inequality in Standard F :

$$-a_{i1}x_1 - a_{i2}x_2 - \dots - a_{in}x_n \geq (\leq) (=) -b_i, \quad -b_i > 0.$$

Note that the set of points (x_1, x_2, \dots, x_n) satisfying the first inequality coincides with the set of points (x_1, x_2, \dots, x_n) satisfying the second inequality. Hence, if the given LPP has all its variables non-negatively sign restricted, the the given LPP and its Standard Form have the same FR with the same extreme (corner) points, the same set of optimal solutions and obviously, the same objective function. In view of this both: the given LPP and its Standard Form coincide.

Example 1. Given the LPP

$$\begin{array}{ll}
 \max z = 7x_1 + 3x_2 - 2x_3 & \\
 \text{s.t.} & \\
 5x_1 - 3x_2 + x_3 \geq -8 & \\
 -3x_1 + 5x_2 - 3x_3 \geq 5 & \\
 6x_1 - 3x_2 - 7x_3 \leq -1 & \\
 -x_1 + x_2 + x_3 = -5 & \\
 \hline
 x_1 \geq 0, x_2 \text{ urs}, x_3 \geq 0. &
 \end{array}$$

Convert the LPP in **Standard Form**.

Solution. The first constraint is not in Standard Form because the right-hand side is negative. We multiply both sides of the first constraint by -1 in order to put this constraint in Standard Form:

$$5x_1 - 3x_2 + x_3 \geq -8 \quad \Rightarrow \quad -5x_1 + 3x_2 - x_3 \leq 8 \quad \text{Standard Form.}$$

The second constraint is in Standard Form.

The third constraint:

$$6x_1 - 3x_2 - 7x_3 \leq -1 \quad \Rightarrow \quad -6x_1 + 3x_2 + 7x_3 \geq 1 \quad \text{Standard Form.}$$

The fourth constraint:

$$-x_1 + x_2 + x_3 = -5 \quad \Rightarrow \quad x_1 - x_2 - x_3 = 5 \quad \text{Standard Form.}$$

The decision variables x_1 and x_3 are non-negative in sign hence, they are in Standard Form. However, the decision variable x_2 is urs and it is not in Standard Form. then using the trivial fact that each number can be represented as a difference of two non-negative numbers we have:

$$x_2 = x_2' - x_2'', \quad x_2' \geq 0, \quad x_2'' \geq 0.$$

Then the Standard Form of the given LPP is the following:

$$\begin{array}{ll}
 \max z = 7x_1 + 3x_2' - 3x_2'' - 2x_3 & \\
 \text{s.t.} & \\
 -5x_1 + 3x_2' - 3x_2'' - x_3 \leq 8 & \\
 -3x_1 + 5x_2' - 5x_2'' - 3x_3 \geq 5 & \\
 -6x_1 + 3x_2' - 3x_2'' + 7x_3 \geq 1 & \\
 x_1 - x_2' + x_2'' - x_3 = 5 & \\
 \hline
 x_1 \geq 0, x_2' \geq 0, x_2'' \geq 0, x_3 \geq 0. &
 \end{array}$$

Note that the given LPP and its Standard Form do not coincide because some of the variables of the given LPP are urs.

Example 2. Given the LPP

$$\begin{array}{l} \min z = 3x_1 - 2x_2 + 5x_3 \\ \text{s.t.} \\ -2x_1 + x_2 + 3x_3 \leq 15 \\ 3x_1 - 5x_2 - 7x_3 = -30 \\ -7x_1 + 3x_2 + 6x_3 \geq -20 \\ \hline x_1 \text{ urs, } x_2 \geq 0, x_3 \text{ urs.} \end{array}$$

Convert the LPP in **Standard Form**.

Solution.

$$\begin{array}{l} \min z = 3x_1' - 3x_1'' - 2x_2 + 5x_3' - 5x_3'' \\ \text{s.t.} \\ -2x_1' + 2x_1'' + x_2 + 3x_3' - 3x_3'' \leq 15 \\ -3x_1' + 3x_1'' + 5x_2 + 7x_3' - 7x_3'' = 30 \\ 7x_1' - 7x_1'' - 3x_2 - 6x_3' + 6x_3'' \leq 20 \\ \hline x_1' \geq 0, x_1'' \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0. \end{array}$$

Note that the given LPP and its Standard Form do not coincide because some of the variables of the given LPP are urs.

Remark. If in a given n -variable, m -constraint LPP all variables are non-negatively sign restricted, i.e., $x_i \geq 0, i = 1, 2, \dots, n$, then the given LPP and its Standard Form (the corresponding Standard LPP) coincide: Both, the given LPP and its Standard Form have the same feasible region (FR), the same extreme (corner) points, the same objective function, and the same set of optimal solutions.

Canonical Form of a Linear Programming Problem.

We have seen that a given LPP can have both equality and inequality constraints with right-hand sides that are either non-negative or negative. It also can have variables that are non-negative as well as unrestricted in sign (urs) variables.

Canonical Form of an LPP. A given LPP is in Canonical Form if:

- (1) All constraints of the LPP are equality constraints.
- (2) All variables of the LPP are non-negative.
- (3) The right-hand sides of all constraints of the LPP are non-negative.

The Standard Form of an LPP satisfies the conditions (2) and (3). Hence, first converting the LPP in Standard Form and after making all constraints equalities, will convert the given LPP in Canonical Form.

Example 3. Given the LPP

$$\begin{array}{l} \max z = 7x_1 + 3x_2 - 2x_3 \\ \text{s.t.} \\ 5x_1 - 3x_2 + x_3 \geq -8 \\ -3x_1 + 5x_2 - 3x_3 \geq 5 \\ 6x_1 - 3x_2 - 7x_3 \leq -1 \\ -x_1 + x_2 + x_3 = -5 \\ \hline x_1 \geq 0, x_2 \text{ urs}, x_3 \geq 0. \end{array}$$

Convert the LPP in **Canonical Form**.

Solution. Note that the given LPP and its Standard Form do not coincide because some of the variables of the given LPP are urs. We convert the LPP in Standard Form:

$$\begin{array}{l} \max z = 7x_1 + 3x_2' - 3x_2'' - 2x_3 \\ \text{s.t.} \\ -5x_1 + 3x_2' - 3x_2'' - x_3 \leq 8 \\ -3x_1 + 5x_2' - 5x_2'' - 3x_3 \geq 5 \\ -6x_1 + 3x_2' - 3x_2'' + 7x_3 \geq 1 \\ x_1 - x_2' + x_2'' - x_3 = 5 \\ \hline x_1 \geq 0, x_2' \geq 0, x_2'' \geq 0, x_3 \geq 0. \end{array}$$

Next, we convert the Standard Form of the LPP (the corresponding standard LPP) in Canonical Form (canonical LPP) by using two types of additional variables: For \leq constraints we use **slack variables** denoted by s_i , $s_i \geq 0$, where i is the number of the constraint. For \geq constraints we use **excess variables** denoted by e_j , $e_j \geq 0$, where j is the number of the constraint.

Consider the first constraint of the Standard Form of the LPP:

$$-5x_1 + 3x_2' - 3x_2'' - x_3 \leq 8 \quad \Rightarrow \quad -5x_1 + 3x_2' - 3x_2'' - x_3 + s_1 = 8 \quad \text{Canonical Form}$$

Consider the second constraint of the Standard Form of the LPP:

$$-3x_1 + 5x_2' - 5x_2'' - 3x_3 \geq 5 \quad \Rightarrow \quad -3x_1 + 5x_2' - 5x_2'' - 3x_3 - e_2 = 5 \quad \text{Canonical Form}$$

Consider the third constraint of the Standard Form of the LPP:

$$-6x_1 + 3x_2' - 3x_2'' + 7x_3 \geq 1 \quad \Rightarrow \quad -6x_1 + 3x_2' - 3x_2'' + 7x_3 - e_3 = 1 \quad \text{Canonical Form}$$

The fourth constraint is an equality constraint and it is in Canonical Form.

Finally, the **Canonical Form** of the LPP is the following:

$$\begin{array}{l} \max z = 7x_1 + 3x_2' - 3x_2'' - 2x_3 \\ \text{s.t.} \\ -5x_1 + 3x_2' - 3x_2'' - x_3 + s_1 = 8 \\ -3x_1 + 5x_2' - 5x_2'' - 3x_3 - e_2 = 5 \\ -6x_1 + 3x_2' - 3x_2'' + 7x_3 - e_3 = 1 \\ x_1 - x_2' + x_2'' - x_3 = 5 \\ \hline x_1 \geq 0, x_2' \geq 0, x_2'' \geq 0, x_3 \geq 0, s_1 \geq 0, e_2 \geq 0, e_3 \geq 0. \end{array}$$

Example 4. Given the LPP

$$\begin{array}{l} \min z = 3x_1 - 2x_2 + 5x_3 \\ \text{s.t.} \\ -2x_1 + x_2 + 3x_3 \leq 15 \\ 3x_1 - 5x_2 - 7x_3 = -30 \\ -7x_1 + 3x_2 + 6x_3 \geq -20 \\ \hline x_1 \text{ urs}, x_2 \geq 0, x_3 \text{ urs}. \end{array}$$

Convert the LPP in **Canonical Form**.

Example 5. Given Giapetto LPP :

$$\begin{array}{ll}
 \max z = 3x_1 + 2x_2 & \\
 \text{s.t.} & \\
 2x_1 + x_2 \leq 100 & \\
 x_1 + x_2 \leq 80 & \\
 x_1 \leq 40 & \\
 \hline
 x_1 \geq 0, x_2 \geq 0. &
 \end{array}$$

(a) Convert the given LPP in Standard Form. In other words, find the corresponding Standard LPP to the given LPP.

(b) Convert the given LPP in Canonical Form. In other words, find the Canonical LPP corresponding to the given LPP.

(c) Write the Canonical LPP obtained in (b) in matrix form.

Solution. (a) The right-hand sides of the constraints in the given LPP are non-negative and the variables of the given LPP are non-negative in sign. Hence, the given LPP is in Standard Form.

Canonical form of the given LPP (the Canonical LPP corresponding to the given LPP):

$$\begin{array}{ll}
 \max z = 3x_1 + 2x_2 & \\
 \text{s.t.} & \\
 2x_1 + x_2 + s_1 = 100 & \\
 x_1 + x_2 + s_2 = 80 & \\
 x_1 + s_3 = 40 & \\
 \hline
 x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0. &
 \end{array}$$

We have used three slack variables s_1, s_2, s_3 because the three constraints in the Standard Form of the given LPP are \leq constraints.

(c) Matrix form of the corresponding Canonical LPP:

Objective function:

$$\max z = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Constraints:

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 80 \\ 40 \end{bmatrix}$$

Sign restrictions:

$$\begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 6. Given Dorian Auto LPP:

$$\begin{array}{ll} \min & z = 50x_1 + 100x_2 \\ \text{s.t.} & \\ & 7x_1 + 2x_2 \geq 28 \\ & 2x_1 + 12x_2 \geq 24 \\ \hline & x_1 \geq 0, x_2 \geq 0. \end{array}$$

(a) Convert the given LPP in Standard Form. In other words, find the corresponding Standard LPP to the given LPP.

(b) Convert the given LPP in Canonical Form. In other words, find the Canonical LPP corresponding to the given LPP.

(c) Write the Canonical LPP obtained in (b) in matrix form.

Solution. (a) The right-hand sides of the constraints in the given LPP are non-negative and the variables of the given LPP are non-negative in sign. Hence, the given LPP is in Standard Form.

(b) Canonical form of the given LPP (the Canonical LPP corresponding to the given LPP):

$$\begin{array}{ll} \min & z = 50x_1 + 100x_2 \\ \text{s.t.} & \\ & 7x_1 + 2x_2 - e_1 = 28 \\ & 2x_1 + 12x_2 - e_2 = 24 \\ \hline & x_1 \geq 0, x_2 \geq 0, e_1 \geq 0, e_2 \geq 0. \end{array}$$

We have used two excess variables e_1, e_2 because the two constraints in the standard form of the given LPP are \geq constraints.

Matrix form of the corresponding Canonical LPP:

Objective function:

$$\min z = \begin{bmatrix} 50 & 100 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}$$

Constraints:

$$\begin{bmatrix} 7 & 2 & -1 & 0 \\ 2 & 12 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 28 \\ 24 \end{bmatrix}$$

Sign restrictions:

$$\begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix} \succeq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 7. Consider the LPP:

$$\begin{array}{l}
 \min z = 50x_1 + 20x_2 + 30x_3 + 80x_4 \\
 \text{s.t.} \\
 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \\
 -3x_1 - 2x_2 \leq -6 \\
 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \\
 -2x_1 - 4x_2 - x_3 - 5x_4 \leq -8 \\
 \hline
 x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.
 \end{array}$$

(a) Convert the given LPP in Standard Form. In other words, find the corresponding Standard LPP to the given LPP.

(b) Convert the given LPP in Canonical Form. In other words, find the Canonical LPP corresponding to the given LPP.

(c) Write the Canonical LPP obtained in (b) in matrix form.

Solution. (a) First, all variables in the given LPP are non-negative sign restricted and by Proposition 2.1, pp.61-62, the given LPP and its Standard form coincide: Both, the given LPP and its Standard Form have the same FR, the same extreme (corner) points, the same objective function, and the same set of optimal solutions (if any).

The right-hand sides of the second and the fourth constraints of the given LPP are < 0 . We multiply the second and the fourth constraint by -1 in order to get the Standard Form:

$$\begin{array}{l}
 \min z = 50x_1 + 20x_2 + 30x_3 + 80x_4 \\
 \text{s.t.} \\
 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \\
 3x_1 + 2x_2 \geq 6 \\
 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \\
 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \\
 \hline
 x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.
 \end{array}$$

(b) Canonical form of the given LPP (the Canonical LPP corresponding to the given

LPP):

$$\begin{aligned} \min z &= 50x_1 + 20x_2 + 30x_3 + 80x_4 \\ \text{s.t.} \\ 400x_1 + 200x_2 + 150x_3 + 500x_4 - e_1 &= 500 \\ 3x_1 + 2x_2 - e_2 &= 6 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 - e_3 &= 10 \\ 2x_1 + 4x_2 + x_3 + 5x_4 - e_4 &= 8 \\ \hline x_1, x_2, x_3, x_4, e_1, e_2, e_3, e_4 &\geq 0. \end{aligned}$$

We have used four excess variables e_1, e_2, e_3, e_4 because the four constraints in the standard form of the given LPP are \geq constraints.

(c) Matrix form of the corresponding Canonical LPP:

Objective function:

$$\min z = \begin{bmatrix} 50 & 20 & 30 & 80 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

Constraints:

$$\begin{bmatrix} 400 & 200 & 150 & 500 & -1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 2 & 2 & 4 & 4 & 0 & 0 & -1 & 0 \\ 2 & 4 & 1 & 5 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

Sign restrictions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 8. Consider the LPP:

$$\begin{array}{ll} \max z = 30x_1 + 100x_2 & \\ \text{s.t.} & \\ x_1 + x_2 \leq 7 & \\ 4x_1 + 10x_2 \leq 40 & \\ -10x_1 \leq -50 & \\ \hline x_1 \geq 0, x_2 \geq 0. & \end{array}$$

Convert the given LPP in Canonical Form. In other words, find the Canonical LPP corresponding to the given LPP.

Solution. First, all variables in the given LPP are non-negative sign restricted hence, it coincides with its Standard Form (see Proposition 2.1 on pp. 61 - 62).

We rewrite all constraints in Standard Form and next convert the Standard Form in Canonical Form:

$$\begin{array}{ll} \max z = 30x_1 + 100x_2 & \\ \text{s.t.} & \\ x_1 + x_2 + s_1 = 7 & \\ 4x_1 + 10x_2 + s_2 = 40 & \\ 10x_1 - e_3 = 50 & \\ \hline x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, e_3 \geq 0. & \end{array}$$

Matrix Form of a Canonical n -variable, m -constraint LPP (an n -variable with m constraints LPP in Canonical Form):

$$\begin{aligned} \max(\min) \ z = \ [\ c_1 \ c_2 \ \cdots \ c_n \] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ \text{s.t.} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\ \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

where $b_1 \geq 0, b_2 \geq 0, \dots, b_m \geq 0$.

Denote:

$$\begin{aligned} \mathbf{c}^t &= \mathbf{c}_{1 \times n}^t = [c_1 \ c_2 \ \cdots \ c_n], \\ \mathbf{x}^t &= \mathbf{x}_{1 \times n}^t = \mathbf{x}_{1 \times n}^t = [x_1 \ x_2 \ \cdots \ x_n], \\ \mathbf{b}^t &= \mathbf{b}_{1 \times m}^t = [b_1 \ b_2 \ \cdots \ b_m], \\ \mathbf{0}^t &= \mathbf{0}_{1 \times n}^t = [0 \ 0 \ \cdots \ 0], \quad (n \text{ zeros}), \end{aligned}$$

where the upper index t means a transpose of a vector.

$$\mathbf{A} = A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

By using the above notations, an n -variable Canonical LPP with m constraints can be written in the following more concise matrix form:

$$\max(\min) \ z = \mathbf{c}^t \mathbf{x} \quad (1)$$

s.t.

$$\mathbf{A} \mathbf{x} = \mathbf{b} \quad (2)$$

$$\mathbf{x} \geq \mathbf{0} \quad (3)$$

The components of the vector \mathbf{c} are the objective function coefficients; the matrix \mathbf{A} is called a constraints' coefficients matrix or simply a constraint matrix.

Summary. Converting a given LPP in Canonical Form.

First, we convert the given LPP in Standard Form:

Step 1. In order to put all constraints in Standard Form, all constraints for which the right-hand side is negative should be multiplied by (-1).

Step 2. If a variable x_i is unrestricted in sign (urs), replace x_i in both, the objective function and the constraints by $x'_i - x''_i$, where $x'_i \geq 0$ and $x''_i \geq 0$.

Next, convert the Standard Form of the given LPP in Canonical Form:

Step 3. If the i_{th} constraint is \leq constraint, add a slack variable s_i and the sign restriction $s_i \geq 0$.

If the i_{th} constraint is \geq constraint, subtract an excess variable e_i and add the sign restriction $e_i \geq 0$.

If the i_{th} constraint is $=$ constraint (with non-negative right-hand side), it stays unchanged.

Basic Solutions (BS) of a Canonical LPP. Basic Feasible Solutions (BFS) of a Canonical LPP. Algebraic Computing of the Extreme (Corner) Points of the Feasible Region of a given LPP.

We develop an algebraic method to compute the extreme (corner) points of the FR of a given LPP. The algebraic method described in the present section is based on the notions of Basic Solutions (BS) and Basic Feasible Solutions (BFS) of a Canonical LPP (a Canonical Form of a given LPP).

Given an n -variable LPP with m constraints in Canonical Form (1),(2),(3), where \mathbf{A} is the $m \times n$ constraints'coefficient matrix, $\mathbf{c} \in R^n$, $\mathbf{x} \in R^n$, and $\mathbf{b} \in R^m$. Denote the columns of the matrix \mathbf{A} by $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$. Then (2) can be written as

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \dots + x_n\mathbf{A}_n = \mathbf{b}. \quad (4)$$

We make two assumptions about the constraint matrix \mathbf{A} . We assume that $m \leq n$ and there are m columns of \mathbf{A} that are linearly independent. i.e., the rank of \mathbf{A} is m . The set of m columns that are linearly independent, form a basis in R^m . We can always renumber the columns of \mathbf{A} by reordering the components of \mathbf{x} such that the first m columns of \mathbf{A} are linearly independent. Denote by S the FR of the Canonical LPP (1),(2),(3).

Theorem 1. *Suppose that the first m columns of the constraint matrix \mathbf{A} are linearly independent and suppose that for the numbers x_1, x_2, \dots, x_m we have*

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \dots + x_m\mathbf{A}_m = \mathbf{b}, \quad (5)$$

where $x_i \geq 0, i = 1, 2, \dots, m$. Then the n -dimensional point

$$\mathbf{x} = (x_1, x_2, \dots, x_m, 0, 0, \dots, 0)$$

is an extreme (corner) point of the feasible region S of the Canonical LPP (1),(2),(3).

Proof. We have $\mathbf{x} \geq \mathbf{0}$ in the statement of the theorem. Equation (5) is in fact $\mathbf{Ax} = \mathbf{b}$ because the last $n - m$ components of \mathbf{x} are zeros. Hence, \mathbf{x} is a feasible solution of the LPP (1),(2),(3); in other words, \mathbf{x} belongs to the FR of the Canonical LPP (1),(2),(3).

Assume to the contrary, that \mathbf{x} is not an extreme (corner) point of the feasible region S . The \mathbf{x} is an interior point of a line segment in S that is:

There are two n -dimensional points in the feasible region S (the end points of the line segment)

$$\mathbf{v} = (v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n)$$

and

$$\mathbf{w} = (w_1, w_2, \dots, w_m, w_{m+1}, \dots, w_n)$$

in both different from \mathbf{x} and a positive number α , $0 < \alpha < 1$ such that

$$\mathbf{x} = \alpha \mathbf{v} + (1 - \alpha) \mathbf{w}.$$

If we compare the components in the above vector equation we obtain:

$$x_i = \alpha v_i + (1 - \alpha) w_i, \quad i = 1, 2, \dots, m.$$

$$0 = \alpha v_i + (1 - \alpha) w_i, \quad i = m + 1, m + 2, \dots, n.$$

Since $v_i \geq 0$ and $w_i \geq 0$ for $i = 1, 2, \dots, n$ and $\alpha > 0$, $(1 - \alpha) > 0$ we conclude that $v_i = w_i = 0$, $i = m + 1, m + 2, \dots, n$.

In addition, \mathbf{v} and \mathbf{w} are feasible solutions (belong to the feasible region S) we have that

$$\mathbf{A}\mathbf{v} = \mathbf{b} \quad \text{and} \quad \mathbf{A}\mathbf{w} = \mathbf{b}$$

and because the last $n - m$ components of \mathbf{v} and \mathbf{w} are zeros the equation $\mathbf{A}\mathbf{v} = \mathbf{b}$ can be written as it follows:

$$v_1 \mathbf{A}_1 + v_2 \mathbf{A}_2 + \dots + v_m \mathbf{A}_m = \mathbf{b},$$

Subtracting the above equation from (5) we obtain:

$$(x_1 - v_1) \mathbf{A}_1 + (x_2 - v_2) \mathbf{A}_2 + \dots + (x_m - v_m) \mathbf{A}_m = \mathbf{0},$$

and in view of the fact that $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_m$ are m vectors in R^m that are linearly independent we conclude that

$$x_1 - v_1 = x_2 - v_2 = \dots = x_m - v_m = 0 \quad \Rightarrow \quad \mathbf{x} = \mathbf{v}.$$

We arrive at a contradiction because we have assumed that $\mathbf{x} \neq \mathbf{v}$ and $\mathbf{x} \neq \mathbf{w}$. Hence, \mathbf{x} is an extreme (corner) point of the feasible region S of the Canonical LPP (1),(2),(3).

Theorem 2. *If*

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

is an extreme (corner) points of the feasible region S of the Canonical LPP (1),(2),(3) then the columns of the constraint matrix \mathbf{A} that correspond to positive values x_i , $i = 1, 2, \dots, n$ form a linearly independent set of vectors in R^m .

Proof. Without any restriction and by using an appropriate renumbering of the variables x_1, x_2, \dots, x_n we can suppose that the first r of the variables are positive and the rest are zero. Then, the equation (4) can be written as

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \dots + x_r\mathbf{A}_r = \mathbf{b}.$$

Suppose to the contrary, that the vectors

$$\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_r$$

are linearly dependent. Then,

$$c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_r\mathbf{A}_r = \mathbf{0}$$

for some numbers c_i , $i = 1, 2, \dots, r$ such that at least one of these numbers is nonzero. Then for $0 < d < \min_i[x_j/|c_j|]$, $c_j \neq 0$ we have

$$(x_1 + dc_1)\mathbf{A}_1 + (x_2 + dc_2)\mathbf{A}_2 + \dots + (x_r + dc_r)\mathbf{A}_r = \mathbf{b}$$

$$(x_1 - dc_1)\mathbf{A}_1 + (x_2 - dc_2)\mathbf{A}_2 + \dots + (x_r - dc_r)\mathbf{A}_r = \mathbf{b}$$

with $x_i \pm dc_i > 0$, $i = 1, 2, \dots, r$. Then

$$\mathbf{v} = (x_1 + dc_1, x_2 + dc_2, \dots, x_r + dc_r, 0, 0, \dots, 0)$$

$$\mathbf{w} = (x_1 - dc_1, x_2 - dc_2, \dots, x_r - dc_r, 0, 0, \dots, 0)$$

are feasible solutions of the Canonical LPP (1),(2),(3) such that

$$\mathbf{x} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$$

but this contradicts the hypothesis that \mathbf{x} is an extreme corner) point of the feasible region S of the Canonical LPP (1),(2),(3) Thus our assumption that the first r column-vectors of \mathbf{A} are linearly dependent is false hence, these r column-vectors are linearly independent.

In view of this, $r \leq m$.

Corollary 1. *If*

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

is an extreme (corner) point for the feasible region S of the Canonical LPP (1), (2), (3) and

$$x_{i_1}, x_{i_2}, \dots, x_{i_r}$$

are the r positive components of \mathbf{x} , then:

(1) *We have $r \leq m$ which means that at most m components of any extreme (corner) point of the feasible region S of the Canonical LPP (1),(2),(3) can be positive. The rest, that are at least $n - m$ must be zero.*

(2) *If $r < m$, then the set of r column-vectors*

$$\mathbf{A}_{i_1}, \mathbf{A}_{i_2}, \dots, \mathbf{A}_{i_r}$$

can be extended to a set of m linearly independent vectors in R^m by adjoining a suitably set of $m - r$ columns of \mathbf{A} .

Basic Solution of a Canonical LPP (1), (2), (3). A basic solution (BS) we obtain by setting $n - m$ of the variables x_1, x_2, \dots, x_n equal to zero and solving for the remaining m variables. **Thus, for a Canonical LPP with n variables and m constraints (1), (2), (3) we have a finite number of basic solutions. More precisely we have at most**

$$\binom{n}{n-m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

basic solutions for a Canonical LPP (1), (2), (3) with n variables and m constraints.

Basic Feasible Solution of the Canonical LPP (1), (2), (3). A basic feasible solution (BFS) is a basic solution that is a feasible solution, also. In other words, a BFS is a BS in which, all variables are non-negative in sign. Remind that the set of feasible solution of a given LPP is the set of all points in the feasible region of the LPP. **Thus, for a Canonical LPP with n variables and m constraints (1), (2), (3) we have a finite number of basic feasible solutions. More precisely, we have at most**

$$\binom{n}{n-m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

basic feasible solutions for a Canonical LPP (1), (2), (3) with n variables and m constraints.

Remark. Although the term **basic solution (BS)** appears in all the literature about Linear Programming it can be misleading. Each basic solution (BS) is a solution $\mathbf{x}^t = \mathbf{x}_{1 \times n}^t = (x_1, x_2, \dots, x_n)$ to the constraint system:

$$\mathbf{A}_{m \times n} \mathbf{x} = \mathbf{b}$$

such that $n - m$ of the variables x_1, x_2, \dots, x_n are set to zero.

A basic solution does not necessarily satisfy $\mathbf{x} \geq \mathbf{0}$ and therefore it is not necessarily a feasible solution (FS) to the Canonical LPP (1), (2), (3). Thus:

$$(\text{the set of BFS}) \subset (\text{the set of BS})$$

$$(\text{the set of BFS}) \subset (\text{the set of FS})$$

$$(\text{the set of BFS}) = (\text{the set of BS}) \cap (\text{the set of FS}).$$

Basic Variables and Non-basic Variables for a Basic Solution and a Basic Feasible Solution of a Canonical LPP. In each basic solution (BS) or basic feasible solution (BFS) the $n - m$ variables that are set to 0 are called non-basic variables of the BS or the BFS. The other m variables with respect to which we solved the constraint system are called basic variables of the BS or the BFS.

The Relationship between the Extreme (Corner) Points of the Feasible Region of the Standard Form of a given LPP and the Extreme (Corner) Points of the Feasible Region of the Canonical Form of the LPP.

First, let us remind the following trivial but important fact: If in the given LPP all variables are non-negative restricted in sign, then the feasible region (FR) of the given LPP and the feasible region (FR) of its Standard Form (in other words the FR of the corresponding Standard LPP) coincide. **In other words, if in the given LPP all variables are non-negative sign restricted, then both the given LPP and its corresponding Standard LPP have one and the same feasible region.**

Given an LPP. First, we convert the LPP in Standard Form; in other words we construct its corresponding Standard LPP. Next, we convert the Standard Form of the given LPP to a Canonical Form by using additional slack and excess variables. In other words we construct the corresponding Canonical LPP to the given LPP.

Theorem 3. *Given an LPP. Denote the FR of its Standard LPP by S' and the FR of its Canonical LPP by S . Then the following holds: Every extreme (corner) point of S' yields an extreme (corner) point (basic feasible solution) of S when slack and excess variables are added. Conversely, every extreme (corner) point (basic feasible solution) of S , when truncated the slack and the excess variables, yields an extreme (corner) point of S' .*

By Theorems 1 and 2 we conclude that the extreme (corner) points of the feasible region S of the Canonical LPP is in fact the set of all basic feasible solutions of the Canonical LPP.

Problem 1. Consider the LPP:

$$\begin{array}{ll} \max z = x_1 + x_2 & \\ \text{s.t.} & \\ 2x_1 + 3x_2 \leq 6 & \\ \hline x_1 \geq 0, x_2 \geq 0, & \end{array}$$

- (a) Compute the extreme (corner) points of the feasible region of the given LPP.
(b) Solve the given LPP.

Solution. (a) First, the given LPP is in Standard Form. In other words the given LPP coincides with its corresponding Standard LPP. Hence, both the given LPP and its Standard LPP have the same feasible regions with the same extreme (corner) points.

Next, construct the Canonical Form of the given LPP by using slack and excess variables:

$$\begin{array}{ll} \max z = x_1 + x_2 & \\ \text{s.t.} & \\ 2x_1 + 3x_2 + s_1 = 6 & \\ \hline x_1 \geq 0, x_2 \geq 0, s_1 \geq 0. & \end{array}$$

We have $n = 3$ and $m = 1$. In order to compute the basic solutions of the Canonical Form we are to set $n - m = 2$ of the variables equal to zero. We obtain:

$$\begin{array}{ll} x_1 = x_2 = 0 & \Rightarrow s_1 = 6 \\ x_1 = s_1 = 0 & \Rightarrow x_2 = 2 \\ x_2 = s_1 = 0 & \Rightarrow x_1 = 3 \end{array}$$

and the basic solutions of the Canonical LPP are:

$$(x_1 = 0, x_2 = 0, s_1 = 6); (x_1 = 0, s_1 = 0, x_2 = 2); (x_1 = 3, x_2 = 0, s_1 = 0).$$

Next, by choosing from the set of basic solutions those having non-negative coordinates we obtain the set of all basic feasible solutions of the Canonical LPP:

$$(x_1 = 0, x_2 = 0, s_1 = 6); (x_1 = 0, s_1 = 0, x_2 = 2); (x_1 = 3, x_2 = 0, s_1 = 0).$$

By Theorems 1 and 2 the set of all basic feasible solutions is the set of all extreme (corner) points of the feasible region of the Canonical LPP.

Hence, the feasible region of the canonical LPP is a space triangle in the three-dimensional x_1, x_2, s_1 -space with three vertices (corners):

$$(x_1 = 0, x_2 = 0, s_1 = 6); (x_1 = 0, s_1 = 0, x_2 = 2); (x_1 = 3, x_2 = 0, s_1 = 6).$$

Now, truncating the slack variable s_1 we shall obtain the extreme (corner) points of the Standard LPP that are the extreme (corner) points of the given LPP in our case:

$$(x_1 = 0, x_2 = 0); (x_1 = 0, x_2 = 2); (x_1 = 3, x_2 = 0)$$

and from here the feasible region of the given LPP is the planar triangle on the x_1, x_2 -plane with vertices (corners):

$$(x_1 = 0, x_2 = 0); (x_1 = 0, x_2 = 2); (x_1 = 3, x_2 = 0).$$

Remark. Try to solve graphically Problem 1, (a) in order to see better the meaning of Theorems 1, 2 and 3. Truncating geometrically means orthogonal projection on a sub-space. In our case, the orthogonal projection of the FR of the Canonical LPP (it is in the 3-space x_1, x_2, s_1 onto the 2-space x_1, x_2 is the FR of the given LPP.

(b) Obviously, the FR of the given LPP is bounded. By EPT, Part (1):

$$\max\{z(0, 0), z(0, 2), z(3, 0)\} = z(3, 0) = 3$$

and from here, the unique optimal solution of the given LPP is

$$x_1 = 3, x_2 = 0, \quad z_{\max} = 3.$$

Problem 2. Given the LPP:

$$\begin{array}{l} \min z = -x_1 + 3x_2 - 2x_3 \\ \text{s.t.} \\ 3x_1 + 2x_2 + x_3 \geq 6 \\ x_1 + x_2 + x_3 \leq 5 \\ \hline x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array}$$

- (a) Compute the extreme (corner) points of the feasible region of the given LPP.
 (b) Solve the given LPP.

Solution. (a) First, the given LPP is in Standard Form. In other words the given LPP coincides with its corresponding Standard LPP. Hence, both the given LPP and its Standard LPP have the same feasible regions with the same extreme (corner) points.

Next, we construct the Canonical Form of the given LPP by using slack and excess variables. In our case we need one excess variable and one slack variable:

$$\begin{array}{l} \min z = -x_1 + 3x_2 - 2x_3 \\ \text{s.t.} \\ 3x_1 + 2x_2 + x_3 - e_1 = 6 \\ x_1 + x_2 + x_3 + s_2 = 5 \\ \hline x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, e_1 \geq 0, s_2 \geq 0, . \end{array}$$

We have $n = 5$ and $m = 2$. In order to compute the basic solutions of the Canonical Form we are to set $n - m = 3$ of the variables equal to zero. We have to check $\binom{5}{3} = 10$ cases:

$$\begin{array}{l} e_1 = s_2 = x_3 = 0 \quad \Rightarrow \quad x_1 = -4, x_2 = 9 \\ e_1 = x_2 = s_2 = 0 \quad \Rightarrow \quad x_1 = 0.5, x_3 = 4.5; \\ x_2 = x_3 = s_2 = 0 \quad \Rightarrow \quad x_1 = 5, e_1 = 9; \\ \\ x_2 = x_3 = e_1 = 0 \quad \Rightarrow \quad x_1 = 2, s_2 = 3; \\ \\ x_1 = e_1 = s_2 = 0 \quad \Rightarrow \quad x_2 = 1, x_3 = 4; \\ \\ x_1 = x_3 = s_2 = 0 \quad \Rightarrow \quad x_2 = 5, e_1 = 4; \end{array}$$

$$x_1 = x_3 = e_1 = 0 \quad \Rightarrow \quad x_2 = 3, s_2 = 2;$$

$$x_1 = x_2 = s_2 = 0 \quad \Rightarrow \quad x_3 = 5, e_1 = -1;$$

$$x_1 = x_2 = e_1 = 0 \quad \Rightarrow \quad x_3 = 6, s_2 = -1;$$

$$x_1 = x_2 = x_3 = 0 \quad \Rightarrow \quad e_1 = -6, s_2 = 5;$$

and we have obtained 10 basic solutions of the Canonical LPP:

$$(x_1 = -4, x_2 = 9, x_3 = 0, e_1 = 0, s_2 = 0);$$

$$(x_1 = 0.5, x_2 = 0, x_3 = 4.5, e_1 = 0, s_2 = 0);$$

$$(x_1 = 5, x_2 = 0, x_3 = 0, e_1 = 9, s_2 = 0);$$

$$(x_1 = 2, x_2 = 0, x_3 = 0, e_1 = 0, s_2 = 3);$$

$$(x_1 = 0, x_2 = 1, x_3 = 4, e_1 = 0, s_2 = 0);$$

$$(x_1 = 0, x_2 = 5, x_3 = 0, e_1 = 4, s_2 = 0);$$

$$(x_1 = 0, x_2 = 3, x_3 = 0, e_1 = 0, s_2 = 2);$$

$$(x_1 = 0, x_2 = 0, x_3 = 5, e_1 = -1, s_2 = 0);$$

$$(x_1 = 0, x_2 = 0, x_3 = 6, e_1 = 0, s_2 = -1);$$

$$(x_1 = 0, x_2 = 0, x_3 = 0, e_1 = -6, s_2 = 5);$$

Next, by choosing from the set of basic solutions those having non-negative coordinates we obtain the set of all basic feasible solutions of the Canonical LPP:

$$(x_1 = 0.5, x_2 = 0, x_3 = 4.5, e_1 = 0, s_2 = 0);$$

$$(x_1 = 5, x_2 = 0, x_3 = 0, e_1 = 9, s_2 = 0);$$

$$(x_1 = 2, x_2 = 0, x_3 = 0, e_1 = 0, s_2 = 3);$$

$$(x_1 = 0, x_2 = 1, x_3 = 4, e_1 = 0, s_2 = 0);$$

$$(x_1 = 0, x_2 = 5, x_3 = 0, e_1 = 4, s_2 = 0);$$

$$(x_1 = 0, x_2 = 3, x_3 = 0, e_1 = 0, s_2 = 2).$$

Hence, we have obtained 6 basic feasible solutions. By Theorems 1 and 2 the set of all basic feasible solutions is the set of all extreme (corner) points of the feasible region of the Canonical LPP.

Hence, the feasible region of the Canonical LPP is a convex set in the 5 dimensional space x_1, x_2, x_3, e_1, s_2 with extreme (corner) points the above 6 basic feasible solutions of the Canonical LPP.

Now, truncating the excess and the slack variables e_1, s_2 we shall obtain the extreme (corner) points of the Standard LPP that are the extreme (corner) points of the Standard LPP and of the given LPP in our case:

$$(x_1 = 0.5, x_2 = 0, x_3 = 4.5);$$

$$(x_1 = 5, x_2 = 0, x_3 = 0);$$

$$(x_1 = 2, x_2 = 0, x_3 = 0);$$

$$(x_1 = 0, x_2 = 1, x_3 = 4);$$

$$(x_1 = 0, x_2 = 5, x_3 = 0);$$

$$(x_1 = 0, x_2 = 3, x_3 = 0).$$

and from here the feasible region of the given LPP is a convex solid (convex polyhedron) in the 3-space x_1, x_2, x_3 with 6 vertices (corners):

$$(x_1 = 0.5, x_2 = 0, x_3 = 4.5);$$

$$(x_1 = 5, x_2 = 0, x_3 = 0);$$

$$(x_1 = 2, x_2 = 0, x_3 = 0);$$

$$(x_1 = 0, x_2 = 1, x_3 = 4);$$

$$(x_1 = 0, x_2 = 5, x_3 = 0);$$

$$(x_1 = 0, x_2 = 3, x_3 = 0).$$

Note that by using the vertices of the FR and the fact that the FR is a convex set (convex polyhedron) we can sketch the feasible region in the 3-dimensional x_1, x_2, x_3 -space.

(b) Obviously, $(x_1 + x_2 + x_3 \leq 5, x_1, x_2, x_3 \geq 0)$ the FR of the given LPP is bounded. By EPT, part (1):

$$\min\{z(0.5, 0, 4.5), z(5, 0, 0), z(2, 0, 0), z(0, 1, 4), z(0, 5, 0), z(0, 3, 0)\} = z(0.5, 0, 4.5) = -9.5$$

and from here, the unique optimal solution of the given LPP is

$$x_1 = 0.5, x_2 = 0, x_3 = 4.5, \quad z_{\min} = -9.5.$$

By combining the Extreme Point Theorem and Theorems 1, 2, 3, we present an Algebraic Method to solve Linear Programming Problems (see Problems 1, 2, given above):

Step 1. Convert the given LPP in a Standard Form.

Step 2. Convert the Standard Form of the LPP in a Canonical Form.

Step 3. Suppose that the Canonical Form has n variables and m constraints. By setting $n - m$ of the variables equal to zero and solving for the rest m variables; and exhausting all possible cases, we obtain **the set of all Basic Solutions of the Canonical LPP.**

Step 4. We choose from the set of all Basic Solutions (BS) those having non-negative components in order to construct **the set of all Basic Feasible Solutions of the Canonical LPP. The set of all Basic Feasible Solutions of the Canonical LPP is in fact the set of all Extreme (Corner) Points of the feasible region of the Canonical LPP.**

Step 5. Truncate the slack and the excess variables from the extreme (corner) points (the basic feasible solutions) of the feasible region of the Canonical LPP in order to obtain the set of all extreme (corner) points of the feasible region of the Standard LPP.

Step 6. If the feasible region of the Standard LPP is bounded, then apply EPT, part (1) in order to find an optimal solution of the Standard LPP. If the feasible region of the Standard LPP is not bounded then in order to apply EPT, part (2) we have to show first that an optimal solution of the Standard LPP exists, i.e., we have to show first that the Standard LPP is bounded and then to apply EPT, part (2).

Step 7. (a) If in the given LPP all variables are non-negative in sign, then both the given LPP and its Standard Form have one and the same feasible region, one and the the same extreme (corner) points of their FRs, and one and the same optimal solutions. Hence, the optimal solutions of the Standard Form are the optimal solutions of the given LPP.

(b) If some of the variables in the given LPP are urs then the given LPP and its Standard Form will not be the same and they will have different feasible regions and different extreme (corner) points of their FRs. In fact, in this case the given LPP

and its Standard Form will be of different dimensions. However, the given LPP, its Standard Form, and its Canonical Form have the same optimal values of the corresponding objective functions.

The three LPP: The given LPP, its Standard LPP, its Canonical LPP have the same optimal values (MIN or MAX) of the corresponding objective functions. The three LPP: The given LPP, its Standard LPP, its Canonical LPP are either simultaneously bounded or simultaneously unbounded. Then, by using the optimal solutions of the Standard LPP, construct optimal solutions of the given LPP .

Problem 3. Consider Giapetto's LPP:

$$\begin{array}{ll}
 \max & z = 3x_1 + 2x_2 \\
 \text{s.t.} & \\
 & 2x_1 + x_2 \leq 100 \\
 & x_1 + x_2 \leq 80 \\
 & x_1 \leq 40 \\
 \hline
 & x_1 \geq 0, x_2 \geq 0.
 \end{array}$$

- (a) Transform (convert) the given LPP in Standard Form. In other words, find the corresponding Standard LPP to the given LPP.
- (b) Transform (convert) the given LPP in Canonical Form. In other words, find the Canonical LPP corresponding to the given LPP.
- (c) Find all Basic Solutions (BSs) of the Canonical LPP. Point out the basic variables (BVs) and the non-basic variables (NBVs) in each BS.
- (d) Find all Basic Feasible Solutions of the canonical LPP. Point out the BVs and the NBVs of each BFS of the Canonical LPP.
- (e) Compute the extreme (corner) points of the feasible region (FR) of the Canonical LPP.
- (f) Compute the extreme (corner) points of the feasible region (FR) of the given LPP.
- (g) Use Extreme Point Theorem to find an optimal solution to the given LPP. Why is EPT applicable in our case?

Solution. (a) First, the given LPP is in Standard Form. In other words the given LPP coincides with its corresponding Standard LPP. Hence, both the given LPP and its Standard LPP have the same feasible regions with the same extreme (corner) points.

(b) Next, we construct the Canonical Form of the given LPP by using slack and excess variables. In our case we need three slack variables:

$$\begin{array}{ll}
 \max & z = 3x_1 + 2x_2 \\
 \text{s.t.} & \\
 & 2x_1 + x_2 + s_1 = 100 \\
 & x_1 + x_2 + s_2 = 80 \\
 & x_1 + s_3 = 40 \\
 \hline
 & x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.
 \end{array}$$

(c) We have $n = 5$ and $m = 3$. In order to compute the basic solutions of the Canonical Form we are to set $n - m = 5 - 3 = 2$ of the variables equal to zero. In

each BS we have 3 BVs and 2 NBVs. We have to check $\binom{5}{2} = 10$ cases:

$$\begin{aligned}
\text{NBV: } s_1 = s_2 = 0 &\Rightarrow \text{BV; } x_1 = 20, x_2 = 60, s_3 = 20; \\
\text{NBV: } s_1 = s_3 = 0 &\Rightarrow \text{BV; } x_1 = 40, x_2 = 20, s_2 = 20; \\
\text{NBV: } s_1 = x_1 = 0 &\Rightarrow \text{BV; } x_2 = 100, s_2 = -20, s_3 = 40; \\
\text{NBV: } s_1 = x_2 = 0 &\Rightarrow \text{BV; } x_1 = 50, s_2 = 30, s_3 = -10; \\
\text{NBV: } s_2 = s_3 = 0 &\Rightarrow \text{BV; } x_1 = 40, x_2 = 40, s_1 = -20; \\
\text{NBV: } s_2 = x_1 = 0 &\Rightarrow \text{BV; } x_2 = 80, s_1 = 20, s_3 = 40; \\
\text{NBV: } s_2 = x_2 = 0 &\Rightarrow \text{BV; } x_1 = 80, s_1 = -60, s_3 = -40; \\
\text{NBV: } s_3 = x_1 = 0 &\Rightarrow \text{impossible case;} \\
\text{NBV: } s_3 = x_2 = 0 &\Rightarrow \text{BV; } x_1 = 40, s_2 = 40, s_1 = 20; \\
\text{NBV: } x_1 = x_2 = 0 &\Rightarrow \text{BV; } s_1 = 100, s_2 = 80, s_3 = 40;
\end{aligned}$$

and we have obtained $9 < \binom{5}{3}$ basic solutions of the Canonical LPP.

(d) Next, by choosing from the set of basic solutions those having non-negative coordinates we obtain the set of all basic feasible solutions of the Canonical LPP:

$$\text{BFS1: } x_1 = 20, x_2 = 60, s_1 = 0, s_2 = 0, s_3 = 20; \text{ BV: } x_1, x_2, s_3; \text{ NBV: } s_1, s_2.$$

$$\text{BFS2: } x_1 = 40, x_2 = 20, s_1 = 0, s_2 = 20, s_3 = 0 \text{ BV: } x_1, x_2, s_2; \text{ NBV: } s_1, s_3.$$

$$\text{BFS3: } x_1 = 0, x_2 = 80, s_1 = 20, s_2 = 0, s_3 = 40; \text{ BV: } x_1, s_1, s_3; \text{ NBV: } x_1, s_2.$$

$$\text{BFS4: } x_1 = 40, x_2 = 0, s_1 = 20, s_2 = 40, s_3 = 0; \text{ BV: } x_1, s_1, s_2; \text{ NBV: } x_2, s_3.$$

$$\text{BFS5: } x_1 = 0, x_2 = 0, s_1 = 100, s_2 = 80, s_3 = 40; \text{ BV: } s_1, s_2, s_3; \text{ NBV: } x_1, x_2.$$

Hence, we have obtained 5 basic feasible solutions.

(e) By Theorems 2.1 and 2.2 the set of all basic feasible solutions is the set of all extreme (corner) points of the feasible region of the Canonical LPP.

Hence, the feasible region of the Canonical LPP is a convex set (convex polyhedron) in the 5 dimensional space x_1, x_2, s_1, s_2, s_3 with extreme (corner) points the above 5 basic feasible solutions of the Canonical LPP.

(f) Note that the given LPP is in Standard Form. In other words the given LPP and its corresponding Standard LPP have the same feasible region and the same extreme (corner) points. Then, truncating the slack variables s_1, s_2, s_3 we shall obtain the set

of the extreme (corner) points of the Standard LPP that is the set of the extreme (corner) points of the given LPP in our case:

$$EP1: x_1 = 20, x_2 = 60;$$

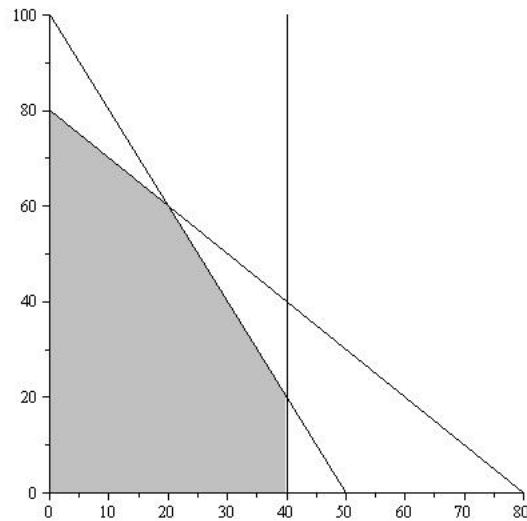
$$EP2: x_1 = 40, x_2 = 20;$$

$$EP3: x_1 = 0, x_2 = 80;$$

$$EP4: x_1 = 40, x_2 = 0;$$

$$EP5: x_1 = 0, x_2 = 0.$$

Note that by using the vertices (the extreme (corner) points) of the FR and the fact that the FR is a convex set (convex polygon) we can sketch the feasible region in the 2-dimensional x_1, x_2 -space (see page 13 for the FR of Giapetto's LPP).



Giapetto's LPP feasible region is the grey shadowed polygon with 5 vertices that are the extreme (corner) points of the FR:

$$EP1(20, 60), EP2(40, 20), EP3(0, 80), EP4(40, 0), EP5(0, 0).$$

(g) Obviously, the FR of the given LPP is bounded. We apply EPT, part (1):

$$\max\{z(EP1), z(EP2), z(EP3), z(EP4), z(EP5)\} = z(EP1) = z(20, 60) = 180$$

and from here, the unique optimal solution of the given LPP is

$$x_1 = 20, x_2 = 60, z_{\max} = 180.$$

Problem 4. Solve the LPP

$$\begin{array}{ll} \min & z = x_1 + x_2 \\ \text{s.t.} & \\ & 5x_1 + x_2 \geq 15 \\ & x_1 + 5x_2 \geq 15 \\ \hline & x_1 \text{ urs, } x_2 \geq 0. \end{array}$$

by using the Linear Algebraic Method based on BFS of the corresponding Canonical LPP.

Solution. Step 1. With $x_1 = x_1' - x_1''$, $x_1' \geq 0$ and $x_1'' \geq 0$ we obtain the Standard Form of the LPP:

$$\begin{array}{ll} \min & z = x_1' - x_1'' + x_2 \\ \text{s.t.} & \\ & 5x_1' - 5x_1'' + x_2 \geq 15 \\ & x_1' - x_1'' + 5x_2 \geq 15 \\ \hline & x_1' \geq 0, x_1'' \geq 0, x_2 \geq 0. \end{array}$$

Step 2. Canonical Form:

$$\begin{array}{ll} \min & z = x_1' - x_1'' + x_2 \\ \text{s.t.} & \\ & 5x_1' - 5x_1'' + x_2 - e_1 = 15 \\ & x_1' - x_1'' + 5x_2 - e_2 = 15 \\ \hline & x_1' \geq 0, x_1'' \geq 0, x_2 \geq 0, e_1 \geq 0, e_2 \geq 0. \end{array}$$

Step 3. $n = 5$, $m = 2$, $n - m = 3$. Each basic solution will have 2 basic variables; and 3 non-basic variables set to zero.

NBV: $x_2 = e_1 = e_2 = 0$; BV: x_1', x_1'' impossible case;

BS1: NBV: $x_1'' = e_1 = e_2 = 0$; BV: $x_1' = 5/2, x_2 = 5/2$;

BS2 NBV: $x_1'' = x_2 = e_2 = 0$; BV: $x_1' = 15, e_1 = 60$;

BS3: NBV: $x_1'' = x_2 = e_1 = 0$ BV: $x_1' = 3, e_2 = -12$;

BS4 NBV: $x_1' = e_1 = e_2 = 0$ BV: $x_1'' = -3, x_2 = 12/5$;

BS5: NBV: $x_1' = x_2 = e_2 = 0$ BV: $x_1'' = -15, e_1 = 60$

BS6 NBV: $x_1' = x_2 = e_1 = 0$ BV: $x_1'' = -3, e_2 = -12$

BS7: NBV: $x'_1 = x''_1 = e_2 = 0$ BV: $x_2 = 3, e_1 = -12$;

BS8 NBV: $x'_1 = x''_1 = e_1 = 0$ BV: $x_2 = 15, e_2 = 60$;

BS9: NBV: $x'_1 = x''_1 = x_2 = 0$ BV: $e_1 = -15, e_2 = -15$;

We have obtained 9 BSs.

Step 4.

BFS1: NBV: $x''_1 = e_1 = e_2 = 0$; BV: $x'_1 = 5/2, x_2 = 5/2$;

BFS2 NBV: $x''_1 = x_2 = e_2 = 0$; BV: $x'_1 = 15, e_1 = 60$;

BFS3 NBV: $x'_1 = x''_1 = e_1 = 0$ BV: $x_2 = 15, e_2 = 60$;

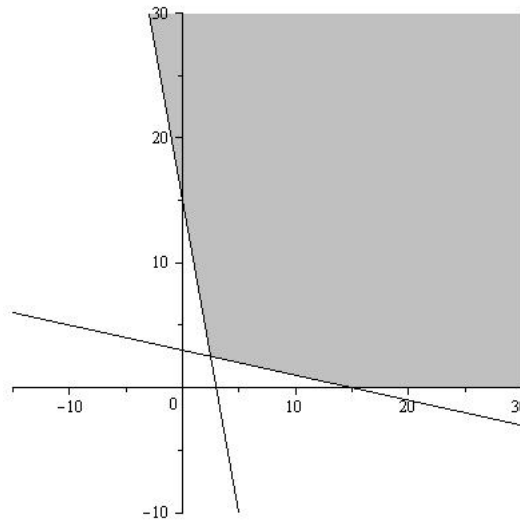
Step 5.

EP1: $x'_1 = 5/2, x''_1 = 0, x_2 = 5/2$;

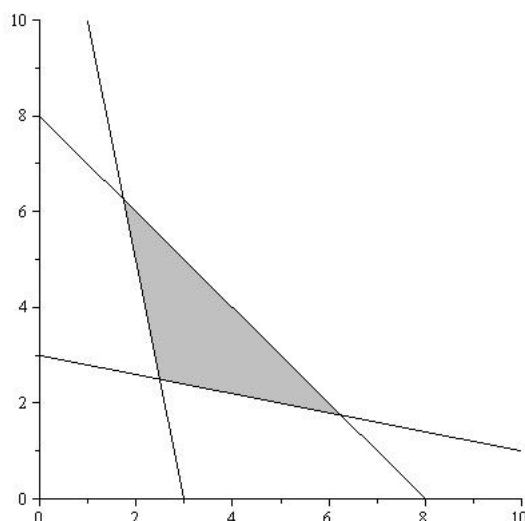
EP2 $x'_1 = 15, x''_1 = 0, x_2 = 0$

EP3 $x'_1 = 0, x''_1 = 0, x_2 = 15$;

Step 6. A bit more careful look shows that if the Standard LPP is unbounded then the given LPP is unbounded, also. However, the point $x_1 = 4, x_2 = 4$ is a feasible solution of the given LPP. Then, by adding the constraint $x_1 + x_2 \leq 8$ to the given LPP we see that the new LPP has the same optimal solutions as the given one and it has a bounded feasible region. From here, the given LPP is bounded and hence, the Standard LPP is also bounded. Here are the feasible regions of the given LPP and the new LPP with the additional constraint $x_1 + x_2 \leq 8$:



The grey shadowed region is the feasible region of the given LPP. The feasible region is unbounded.



The grey shadowed region is the feasible region of the new LPP with the additional constraint $x_1 + x_2 \leq 8$. The feasible region of the new LPP is bounded.

Then, we can apply EPT, part (2) to the Standard LPP:

$$\begin{aligned} z_{\min, \text{standard}} &= \min\{z(5/2, 0, 5/2), z(15, 0, 0), z(0, 0, 15)\} \\ &= \min\{5, 15, 15\} = 5 = z(5/2, 0, 5/2). \end{aligned}$$

Step 7. Taking into account that the given LPP and its Standard LPP have the same min value and the relation

$$x_1 = x_1' - x_1''$$

we conclude that the given LPP has the following optimal solution:

$$x_1 = x_1' - x_1'' = 5/2, \quad x_2 = 5/2, \quad z_{\min} = 5.$$

Problem 5. Solve the LPP

$$\begin{array}{ll} \max & z = 7x_1 + 3x_2 \\ \text{s.t.} & \\ & 5x_1 + 3x_2 \geq 15 \\ & 3x_1 + 5x_2 \geq 15 \\ & x_1 + x_2 = 10 \\ \hline & x_1 \geq 0, x_2 \geq 0. \end{array}$$

by using the Linear Algebra Method.

Solution. Step 1. The given LPP coincides with its Standard LPP:

$$\begin{array}{ll} \max & z = 7x_1 + 3x_2 \\ \text{s.t.} & \\ & 5x_1 + 3x_2 \geq 15 \\ & 3x_1 + 5x_2 \geq 15 \\ & x_1 + x_2 = 10 \\ \hline & x_1 \geq 0, x_2 \geq 0. \end{array}$$

Step 2. Canonical Form:

$$\begin{array}{ll} \max & z = 7x_1 + 3x_2 \\ \text{s.t.} & \\ & 5x_1 + 3x_2 - e_1 = 15 \\ & 3x_1 + 5x_2 - e_2 = 15 \\ & x_1 + x_2 = 10 \\ \hline & x_1 \geq 0, x_2 \geq 0, e_1 \geq 0, e_2 \geq 0. \end{array}$$

Step 3. $n = 4, m = 3, n - m = 1$. Each basic solution will have three basic variables; and one non-basic variable set to zero.

$$\mathbf{BS1:} \quad \text{NBV: } x_1 = 0; \quad \text{BV: } x_2 = 10, e_1 = 15, e_2 = 35;$$

$$\mathbf{BS2:} \quad \text{NBV: } x_2 = 0; \quad \text{BV: } x_1 = 10, e_1 = 35, e_2 = 15;$$

$$\mathbf{BS3} \quad \text{NBV: } e_1 = 0; \quad \text{BV: } x_1 = -15/2, x_2 = 35/2, e_2 = 50;$$

$$\mathbf{BS4:} \quad \text{NBV: } e_2 = 0 \quad \text{BV: } x_1 = 35/2, x_2 = -15/2, e_1 = 50;$$

We have obtained 4 BSs.

Step 4.

$$\mathbf{BFS1:} \quad \text{NBV: } x_1 = 0; \quad \text{BV: } x_2 = 10, e_1 = 15, e_2 = 35;$$

BFS2: NBV: $x_2 = 0$; BV: $x_1 = 10, e_1 = 35, e_2 = 15$;

We have obtained 2 BFSs.

Step 5.

EP1: $x_1 = 0, x_2 = 10$;

EP2 $x_1 = 10, x_2 = 0$;

Steps 6 and 7. The feasible region of the Standard Form of the given LPP is bounded. It has only two extreme (corner) points which means that it is a line segment. In addition, the given LPP and its Standard Form coincide hence, they have the same feasible region, the same extreme (corner) points and the same optimal solutions. We apply EPT, part (1) in order to obtain an optimal solution of the given LPP:

$$z_{\max} = \max\{z(0, 10), z(10, 0)\} = \max\{30, 70\} = 70 = z(10, 0)$$

and in view of this the unique optimal solution of the given LPP is:

$$x_1 = 10, x_2 = 0, z_{\max} = 70.$$

Problem 6. Consider Farmer Jones LPP

$$\begin{array}{ll}
 \max & z = 30x_1 + 100x_2 \\
 \text{s.t.} & \\
 & x_1 + x_2 \leq 7 \\
 & 4x_1 + 10x_2 \leq 40 \\
 & 10x_1 \geq 30 \\
 \hline
 & x_1 \geq 0, x_2 \geq 0.
 \end{array}$$

By using the Linear Algebra Method, solve the LPP.

Solution. Step 1. The given LPP coincides with its Standard LPP:

$$\begin{array}{ll}
 \max & z = 30x_1 + 100x_2 \\
 \text{s.t.} & \\
 & x_1 + x_2 \leq 7 \\
 & 4x_1 + 10x_2 \leq 40 \\
 & 10x_1 \geq 30 \\
 \hline
 & x_1 \geq 0, x_2 \geq 0.
 \end{array}$$

Step 2. Canonical Form:

$$\begin{array}{ll}
 \max & z = 30x_1 + 100x_2 \\
 \text{s.t.} & \\
 & x_1 + x_2 + s_1 = 7 \\
 & 4x_1 + 10x_2 + s_2 = 40 \\
 & 10x_1 - e_3 = 30 \\
 \hline
 & x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, e_3 \geq 0.
 \end{array}$$

Step 3. $n = 5, m = 3, n - m = 2$. Each basic solution will have three basic variables; and two non-basic variables set to zero.

(1) **BS1:** NBV: $e_3 = 0, s_2 = 0$; BV: $x_1 = 3, x_2 = 2.8, s_1 = 1.2$;

BS1: $x_1 = 3, x_2 = 2.8, s_1 = 1.2, s_2 = 0, e_3 = 0, s_2 = 0$;

((2) **BS2:** NBV: $e_3 = 0, s_1 = 0$; BV: $x_1 = 3, x_2 = 4, s_2 = -12$;

BS2: $x_1 = 3, x_2 = 4, s_1 = 0, s_2 = -12, e_3 = 0, s_2 = 0$;

(3) **BS3:** NBV: $e_3 = 0, x_2 = 0$; BV: $x_1 = 3, s_1 = 4, s_2 = 28$;
BS3: $x_1 = 3, x_2 = 0, s_1 = 4, s_2 = 28, e_3 = 0$;

(4) NBV: $e_3 = 0, x_1 = 0$ BV: $x_2, s_{1,2}$ **impossible case**;

(5) **BS4:** NBV: $s_2 = 0, s_1 = 0$; BV: $x_1 = 5, x_2 = 2, e_3 = 20$;
BS4: $x_1 = 5, x_2 = 2, s_1 = 0, s_2 = 0, e_3 = 20$;

(6) **BS5:** NBV: $s_2 = 0, x_1 = 0$; BV: $x_2 = 4, s_1 = 3, e_3 = -30$;
BS5: $x_1 = 0, x_2 = 4, s_1 = 3, s_2 = 0, e_3 = -30$;

(7) **BS6:** NBV: $s_2 = 0, x_2 = 0$; BV: $x_1 = 10, s_1 = -3, e_3 = 70$;
BS6: $x_1 = 10, x_2 = 0, s_1 = -3, s_2 = 0, e_3 = 70$;

(8) **BS7:** NBV: $s_1 = 0, x_1 = 0$; BV: $x_2 = 7, s_2 = -30, e_3 = -30$;
BS7: $x_1 = 0, x_2 = 7, s_1 = 0, s_2 = -30, e_3 = -30$;

(9) **BS8:** NBV: $s_1 = 0, x_2 = 0$; BV: $x_1 = 7, s_2 = 12, e_3 = 40$;
BS8: $x_1 = 7, x_2 = 0, s_1 = 0, s_2 = 12, e_3 = 40$;

(10) **BS9:** NBV: $x_1 = 0, x_2 = 0$; BV: $s_1 = 7, s_2 = 40, e_3 = -30$;
BS9: $x_1 = 0, x_2 = 0, s_1 = 7, s_2 = 40, e_3 = -30$;

We have obtained **9 Basic Solutions**.

Step 4. All Basis Solutions that have only non-negative components will form the set of all Basic Feasible Solutions:

$$\mathbf{BFS1:} \quad \text{NBV: } e_3 = 0, s_2 = 0; \quad \text{BV: } x_1 = 3, x_2 = 2.8, s_1 = 1.2;$$

$$\mathbf{BFS1:} \quad x_1 = 3, x_2 = 2.8, s_1 = 1.2, s_2 = 0, e_3 = 0, s_2 = 0;$$

$$\mathbf{BFS2:} \quad \text{NBV: } e_3 = 0, x_2 = 0; \quad \text{BV: } x_1 = 3, s_1 = 4, s_2 = 28;$$

$$\mathbf{BFS2:} \quad x_1 = 3, x_2 = 0, s_1 = 4, s_2 = 28, e_3 = 0;$$

$$\mathbf{BFS3:} \quad \text{NBV: } s_2 = 0, s_1 = 0; \quad \text{BV: } x_1 = 5, x_2 = 2, e_3 = 20;$$

$$\mathbf{BFS3:} \quad x_1 = 5, x_2 = 2, s_1 = 0, s_2 = 0, e_3 = 20;$$

$$\mathbf{BFS4:} \quad \text{NBV: } s_1 = 0, x_2 = 0; \quad \text{BV: } x_1 = 7, s_2 = 12, e_3 = 40;$$

$$\mathbf{BFS4:} \quad x_1 = 7, x_2 = 0, s_1 = 0, s_2 = 12, e_3 = 40;$$

We have found **4 Basic Feasible Solutions**. These are the extreme (corner) points of the FR of the Canonical LPP.

Step 5. Extreme (corner) points of the Standard LPP:

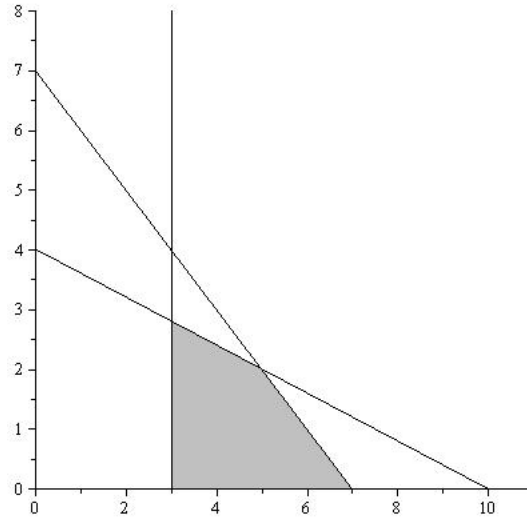
$$\mathbf{EP1:} \quad x_1 = 3, x_2 = 2.8;$$

$$\mathbf{EP2:} \quad x_1 = 3, x_2 = 0;$$

$$\mathbf{EP3:} \quad x_1 = 5, x_2 = 2;$$

$$\mathbf{EP4: } x_1 = 7, x_2 = 0;$$

The given LPP and its Standard Form coincide hence, they have the same feasible region and the same extreme (corner) points. The given LPP is two-variable and we can draw its feasible region in order to verify our result obtained by using the algebraic method described on pp. 85-86:



The grey shadowed region is the feasible region of the given LPP and its Standard Form.
The feasible region is bounded.

Steps 6 and 7. The feasible region of the given LPP is bounded. In addition, the given LPP and its Standard Form coincide hence, they have the same feasible region, the same extreme (corner) points and the same optimal solutions. We apply EPT, part (1) in order to obtain an optimal solution of the given LPP:

$$z_{\max} = \max\{z(3, 2.8), z(3, 0), z(5, 2), z(7, 0)\}$$

$$\max\{370, 90, 350, 210\} = 370 = z(3, 2.8)$$

and in view of this the unique optimal solution of the given LPP is:

$$x_1 = 3, x_2 = 2.8, z_{\max} = 370$$

Problem 7. Consider the subset Ω of all algebraic polynomials $p(x) = a_0 + a_1x + a_2x^2$ of degree ≤ 2 such that $\int_0^1 p(x)dx \leq 1$ and $a_0 \geq 0, a_1 \geq 0, a_2 \geq 0$. Solve the optimization problem:

$$\max_{p \in \Omega} p(1).$$

Solution.

$$\int_0^1 p(x)dx = a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2, \quad p(1) = a_0 + a_1 + a_2$$

and we can write the given optimization problem as a linear programming problem:

$$\begin{aligned} \max \quad & z = a_0 + a_1 + a_2 \\ \text{s.t.} \quad & \\ & a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 \leq 1 \\ & a_0 \geq 0, a_1 \geq 0, a_2 \geq 0. \end{aligned} \quad (*)$$

The Standard Form of the LPP (*) coincides with the given LPP. The Canonical Form of the LPP (*) is:

$$\begin{aligned} \max \quad & z = a_0 + a_1 + a_2 \\ \text{s.t.} \quad & \\ & a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + s_1 = 1 \\ & a_0 \geq 0, a_1 \geq 0, a_2 \geq 0, s_1 \geq 0. \end{aligned}$$

We have $n = 4$ and $m = 1$. Hence, each BS will have 3 NBV variables set to 0 and one BV. First we have to find all Basic Solutions of the Canonical LPP:

$$a_0 = 1, a_1 = 0, a_2 = 0, s_1 = 0;$$

$$a_0 = 0, a_1 = 2, a_2 = 0, s_1 = 0;$$

$$a_0 = 0, a_1 = 0, a_2 = 3, s_1 = 0;$$

$$a_0 = 0, a_1 = 0, a_2 = 0, s_1 = 1.$$

These are the extreme (corner) points of the feasible region of the Canonical LPP. Truncating the slack variable we obtain the extreme (corner) points of the Standard Form of the LPP (*) hence, in our case, of the LPP (*):

$$a_0 = 1, a_1 = 0, a_2 = 0;$$

$$a_0 = 0, a_1 = 2, a_2 = 0;$$

$$a_0 = 0, a_1 = 0, a_2 = 3;$$

$$a_0 = 0, a_1 = 0, a_2 = 0.$$

The LPP (*) has a bounded feasible region and it coincides with its corresponding Standard LPP. Applying EPT, part (1) we obtain:

$$\begin{aligned} z_{\max} &= \max\{z(1, 0, 0), z(0, 2, 0), z(0, 0, 3), z(0, 0, 0)\} \\ &= z(0, 0, 3) = 3 \end{aligned}$$

to conclude that the unique optimal solution of the LPP (*) is:

$$a_0 = 0, a_1 = 0, a_2 = 3 \quad z_{\max} = 3.$$

Hence, the unique optimal solution of the given optimization problem is: $p_*(x) = 3x^2$ with

$$\max_{p \in \Omega} p(1) = p_*(1) = 3.$$