Lecture 1, MAST-224, Winter 2013

We give a description of a Linear Programming Problem by using a real life production process to be optimized. We explain the basic features of a LP by using this example.

What is a Linear Programming Problem?

We start with a real life optimization problem:

LPP Giapetto's Woodcarving , Inc. manufactures 2 types of wooden toys: soldiers and trains.
A soldier sells for 27$ and uses 10$ worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead cost by 14$. A train sells for 21$ and uses 9$ worth of raw materials. Each train built increases Giapetto's variable labor and overhead cost by 10$.
The manufacture of wooden soldiers and trains needs two types of skill labor: carpentry and finishing.
A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hours of finishing labor and 1 hour of carpentry labor.
Each week Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours are available.
Demand for trains is unlimited but at most 40 soldiers are bought each week.
Giapetto's Woodcarving, Inc wants to maximize the weekly profit (revenue) from soldiers and trains.
(a) Formulate a mathematics model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.
(b) Solve graphically the LPP obtained in (a).

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Solution. In developing the Giapetto's model we explore characteristics shared by all linear programming problems.

Decision variables. The decision variables describe completely the decisions to be made (in this case by Giapetto's Woodcarving, Inc. Clearly, Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind we define

x_1 - the number of soldiers produced each week
x_2 - the number of trains produced each week,

So, x_1 and x_2 are our decision variables.

Objective function. In any LP problem, the decision maker wants to maximize (or minimize) some function of the decision variables. The function to be maximize or minimize is called the objective function. For Giapetto's problem we note that fixed costs (such as rent and insurance) do not depend on the values of x_1 and x_2. Thus, Giapetto can concentrate on maximizing (Weekly revenues) - (raw material costs) - (other variable costs)

Giapetto's weekly revenues and costs can be expressed in terms of the decision variables x_1 and x_2.

It would be foolish for Giapetto to manufacture more soldiers than can be sold, so we assume that all toys produced will be sold. Then

Weekly revenues = weekly revenue from soldiers + weekly revenue from trains = $27x_1 + 21x_2$.

Also,

Weekly raw material costs = $10x_1 + 9x_2$

Other weekly variable costs = $14x_1 + 10x_2$

Then Giapetto must maximize

$$(27x_1 + 21x_2) - (10x_1 + 9x_2) - (14x_1 + 10x_2)$$
$$= 3x_1 + 2x_2$$

Another way to see that Giapetto is to maximize $3x_1 + 2x_2$ is the following:

Contribution to profit from soldier = $27 - 10 - 14 = 3$

Contribution to profit from train = $21 - 9 - 10 = 2$

Then, as before

Weekly revenues – Weekly nonfixed costs

$= 3x_1 + 2x_2$ ⇒ The total contribution to profit,

and this is the objective function to be maximized,

Maximize $z = 3x_1 + 2x_2$
and this is our <u>objective function</u> of the <u>decision variables</u> x_1 and x_2,
<u>Objective function coefficients</u> - the coefficients of the variables in the objective function.
In our case the objective function coefficient for x_1 is 3 and the objective function coefficient for x_2 is 2.
In this example (and in many other problems) the objective function coefficient for each variable is simply the contribution of the variable to the company's profit.
<u>Constraints (restrictions)</u>. As x_1 and x_2 increase, Giapetto's objective function grows larger. This means that if Giapetto were free to choose any values for x_1 and x_2, the company could make an arbitrary large profit. However, the values of x_1 and x_2 are limited by the following three restrictions (often called <u>constraints</u>):

- **Constraint 1.** Each week, no more than 100 hours of finishing time may be used.
- **Constraint 2.** Each week, no more than 80 hours of carpentry time may be used.
- **Constraint 3.** Because of limited demand, at most 40 soldiers should be produced each week.
  - The amount of raw material available is assumed unlimited, so no restrictions (constraints) on this.

The next step in formulating a MATHEMATICAL MODEL of Giapetto's problem is to express Constaint 1–3 in terms of the decision variables x_1 and x_2

Constraint 1. $2 \cdot x_1 + 1 \cdot x_2 = 2x_1 + x_2 \leq 100$

Constraint 2. $1 \cdot x_1 + 1 \cdot x_2 = x_1 + x_2 \leq 80$

Constraint 3. $x_1 \leq 40$

Constraints in mathematical form:

$$2x_1 + x_2 \leq 100 \text{ (finishing hours)}$$

⊕ $$x_1 + x_2 \leq 80 \text{ (carpentry hours)}$$

$$x_1 \leq 40 \text{ (limited demand)}$$

The coefficients of the decision variables in the constraints are called technological coefficients. This is because the technological coefficients often reflect the technology used to produce different products.

The RHS - the number on the right-hand side of each constraint is called the constraint's right-hand side. The RHS represents the quantity of a resource that is available

⊕ Sign restrictions on the decision variables

To complete the formulation of a LP problem the following question must be answered: Can the decision variables only take non-negative values, or a decision variable is allowed to take both positive and negative values.

If a decision variable x_i can have only non-negative values, the restriction $x_i \geqslant 0$ - a sign restriction. If x_i can take positive or negative values we call x_i an unrestricted in sign decision variable. So, in our case

$x_1 \geqslant 0$, $x_2 \geqslant 0$ sign constraints.

Summing up we obtain the following mathematical optimization problem (mathematical optimization model):

$$
\begin{cases}
\max\ z = 3x_1 + 2x_2 & \text{(Objective function)} \\
\text{subject to (S.t.)} & \\
2x_1 + x_2 \leq 100 & \text{(finishing constraints)} \\
x_1 + x_2 \leq 80 & \text{(carpentry constraints)} \\
x_1 \leq 40 & \text{(constraint on demand for soldiers)} \\
x_1 \geqslant 0 & \text{(sign restriction)} \\
x_2 \geqslant 0 & \text{(sign restriction)}
\end{cases}
$$

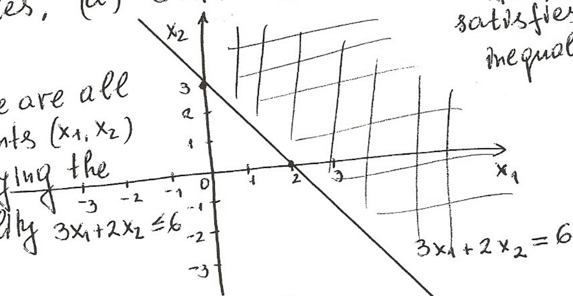
Definition of a LP problem

Definition. $f(x_1, x_2, \ldots, x_n)$ is a linear function if $f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ for some constants $c_1, c_2, \ldots, c_n$.

Definition. Given a linear function $f(x_1, x_2, \ldots, x_n)$ and a number b. The inequalities $f(x_1, \ldots, x_n) \leq b$, $f(x_1, \ldots, x_n) \geq b$ are linear inequalities; $f(x_1, \ldots, x_n) = b$ is a linear equality.

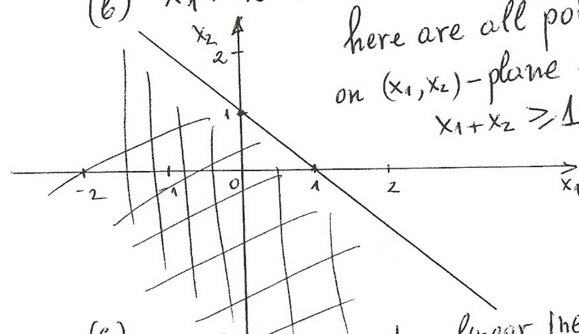
Examples. (a) $3x_1 + 2x_2 \leq 6$, The point $(0,0)$ $x_1 = 0, x_2 = 0$ satisfies this inequality.

here are all points (x_1, x_2) satisfying the inequality $3x_1 + 2x_2 \leq 6$



(b) $x_1 + x_2 \geq 1$

here are all points (x_1, x_2) on (x_1, x_2)-plane satisfying $x_1 + x_2 \geq 1$.



(c) $x_1^2 + x_2^2 \leq 1$ is not a linear inequality.

Definition of LP problem

A LP (linear programming problem) is an optimization problem for which we do the following:

1. We maximize (or minimize) a linear function of the decision variables called the objective function.
2. The values of the decision variables must satisfy a set of constraints which are linear equations or linear inequalities.
3. A sign restriction is associated with each variable. For any variable x_i the sign restriction must be either non-negative $x_i \geqslant 0$ or unrestricted in sign.

p.53 - mistake concerning additivity assumption

line 12 from below ... of the values of the other variables.

- Proportionality Assumption of LP.
- Additivity Assumption of LP.
- Divisibility Assumption (fractional values for the decision variables.
- Integer programming problem.
- Certainty assumption. - true objective and technological coefficients.

Feasible Region and Optimal Solution

Definition. The feasible region for an LP is the set of all points that satisfies all the LP's constraints (restrictions) and sign restrictions.

feasible points and infeasible points

What is feasible? – practicable, possible, manageable, plausible.

Definition. For a max problem, an optimal solution to an LP is a point in the feasible region with the largest value of the objective function.

For a min problem, an optimal solution to an LP is a point in the feasible region with the smallest objective function value.

Feasible Region of an LP

Graphical Solution to a 2-variable LP problem

Giapetto LP is two-variable and it can be solved graphicaly:

The feasible region: all points (x_1, x_2):

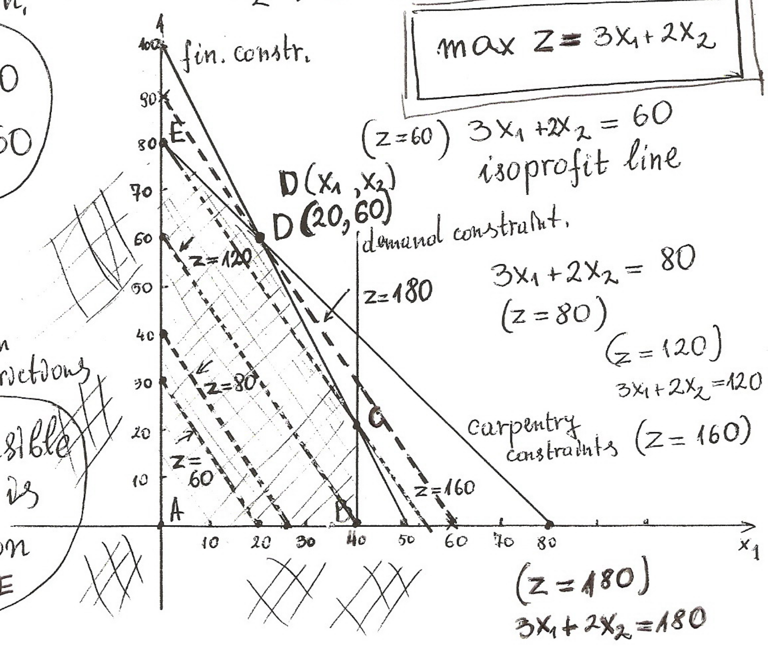
$$\begin{cases} 2x_1 + x_2 \le 100 & \text{(finishing constraint)} \\ x_1 + x_2 \le 80 & \text{(carpentry constraint)} \\ x_1 \le 40 & \text{(demand constraint)} \\ x_1 \ge 0 \\ x_2 \ge 0 \end{cases} \Big\} \text{ sign restrictions}$$

$$\max z = 3x_1 + 2x_2$$

opt. solution.

$x_1 = 20$

$x_2 = 60$



the feasible region is a polygon ABCDE

binding constraint - if the LHS = RHS at the point of optimal solution

non-binding constraint - if the constraint are unequal (strict inequalities) at the point of optimal solution.

Giapetto LP - optimal solution
$x_1 = 20$, $x_2 = 60$ - finishing constraint are binding; carpentry constraint are binding; demand constraint is not binding.

Convex set - if the line-segment jointing any two points is entirely in the set.
Extreme point - each line segment in the set and containing the point has this point as an endpoint. (extreme (corner) point for a given convex set).
Examples of extreme (corner) points
- the vertices of a polygon which is convex

| p.59 - convex polygon - the book |

For LP with two variables
we have one of the following 4 cases:

Case 1. The LP has a unique solution

Case 2. The LP has alternative or multiple optimal solutions: Two or more extreme points are optimal, and the LP have infinite number of optimal solutions.

Case 3. The LP is infeasible: The feasible region is empty - no feasible points.

Case 4. The LP is unbounded:
For max problem there are feasible points with arbitrary large z-values (max problem) or for min problem the are feasible points with arbitrary small z-values,

p. 67 / text book

Remark. An LP with bounded, non-empty feasible region has always at least one optimal solution.

Formulation of the OPTIMIZATION PROBLEM
by using LINDO61

MAX 3S+2T
ST
2S+T<100
S+T<80
S<40
S>0
T>0

Solution of the OPTIMIZATION PROBLEM
by using LINDO61

LP OPTIMUM FOUND AT STEP 3

OBJECTIVE FUNCTION VALUE

1) 180.0000

VARIABLE	VALUE	REDUCED COST
S	20.000000	0.000000
T	60.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	1.000000
3)	0.000000	1.000000
4)	20.000000	0.000000
5)	20.000000	0.000000
6)	60.000000	0.000000

NO. ITERATIONS= 3

The fact that the objective function for a given LP is a linear function of the decision variables has two important applications:

(A) The contribution to the objective function from each decision variable is proportional to the value of the decision variable. For example, the contribution to the objective function in Giapetto's LP (see below) for 4 soldiers is exactly four times the contribution of 1 soldier.

(B) The contribution to the objective function for each decision variable is independent of the other decision variables. For example, (see Giapetto's LP below) no matter what is the value of x_2, the manufacture of x_1 soldiers will always contribute $3x_1$ dollars to the objective function.

Analogously, the fact that each LP constraint is a linear inequality or linear equality has the same 2 applications:
(A) The contribution of each variable to the left-hand side of each constraint is proportional to the value of the variable. For example it takes exactly 3 times more finishing hours to manufacture 3 soldiers than as it does for 1 soldier.
(B) The contribution of a decision variable to the left-hand side of each constraint is independent of the values of the other decision variables. For example, no matter what is the value of x_1 in Giapetto's LP (see below), the manufacture of x_2 trains uses x_2 finishing and x_2 carpentry hours.
The item (A) is called **Proportionality Assumption of the Linear Programming**.
The item (B) is called **Additivity Assumption of the Linear Programming.**
(C) The Divisibility Assumption requires that each decision variable be permitted to assume fractional values up to some limit.
(D) The Certainty Assumption is that each parameter (objective function coefficients, right-hand sides, and technological coefficients) are known with certainty (well investigated).