

9

Probabilities and Reasonable Belief

The probabilities of events, premise truth, and logical strength

Useful insights from the probability calculus

Insight 1: Proportionality

Insight 2: Joint events

Insight 3: Alternative events

Sometimes, assessing the premise truth and the logical strength of an argument requires us to consider carefully the probabilities of events. (Events are just things that happen or occur. The probabilities of events, then, are the chances that things happen or occur.)

Consider, for example, the following argument:

There's a good chance that you'll meet your ex at the party tonight. There's also a good chance that you'll meet your boss at the party. So, there's a good chance that you'll meet both your ex and your boss at the party. And since there's a good chance that you'll meet them both at the party, there's a good chance that you won't enjoy the party.

P1. There's a good chance that you'll meet your ex at the party tonight.

P2. There's a good chance that you'll meet your boss at the party tonight.

C1. There's a good chance both that you'll meet your ex at the party tonight and that you'll meet your boss at the party tonight.

C2. There's a good chance that you won't enjoy the party tonight.

In order to assess the logical strength of the first part of this argument, we have to consider the probability of the event mentioned in C1 in the light of the probabilities of the events mentioned in P1 and P2.

Or take the following argument:

If you get an A on your economics final exam, you'll get an A in the course. Don't worry: you'll get an A on the final exam. So you'll get an A in the course.

P1. If you get an A on your economics final exam, you'll get an A in your economics course.

P2. You'll get an A on your economics final exam.

C. You'll get an A in your economics course.

Even though we can easily know that the logical strength of this argument is good (it's an instance of the valid logical form *Modus Ponens*, after all), we still have to consider the likelihood of the event mentioned in P2 in order to assess the overall soundness of the argument.

What's known as **the probability calculus** provides us with a useful set of rules or insights for figuring out (or "calculating") the probabilities of events. In what follows, we will look at three basic insights that the study of the probability calculus gives us by way of figuring out the probabilities of events, with a view to assessing the premise truth and logical strength of arguments.

The first insight is the most general:

Insight 1: Proportionality. Generally speaking, the probability of an event is the proportion of relevant possibilities that are favorable. I.e., generally speaking,

$$\text{The probability of an event} = \frac{\text{The number of favorable possibilities}}{\text{The number of relevant possibilities}}$$

To understand this insight, we need to get clear about the unusual terms it uses--"relevant possibilities" and "favorable possibilities." When we ask about the probability of an event, we are (at least implicitly) associating the event with two classes of event: a general class and a more specific class. The general class should be relevant to the background scenario we have in mind when asking about the probability of our event. The specific class concerns the part of the general class that is "favorable" to the event, i.e. in which that event more specifically described might occur. To illustrate, suppose that there are four cards, all blank except two, each of which has a blue dot on one side. The dotted cards are placed face down so you can't see them, and all four cards are shuffled randomly. You pick one of them. What's the probability that you pick a dotted card?

In asking this question, we are associating the event it is about with two classes of events: the general class of *your picking one of the cards in the scenario* and the more specific class of *your picking one of the dotted cards in the scenario*.

Similarly, suppose we know that the incidence of a given disease is very low: only 1 in 5,000 people ever get the disease. We then ask ourselves: What's the probability that someone gets the disease? Here too we are associating the event in question with two classes of event: the general class of *someone's being in a group of 5,000 people* and the more specific class of *someone's being in a group of 5,000 people and getting the disease*.

When we talk about **relevant possibilities**, we're talking about all the members of the

general class we associate with an event. (It's helpful to keep in mind that the relevant possibilities should be equiprobable, i.e. each should be as likely to occur as any other, and that the sum of all their probabilities should be 100%.) And when we talk

about **favorable possibilities**, we're talking about all the members of the _____

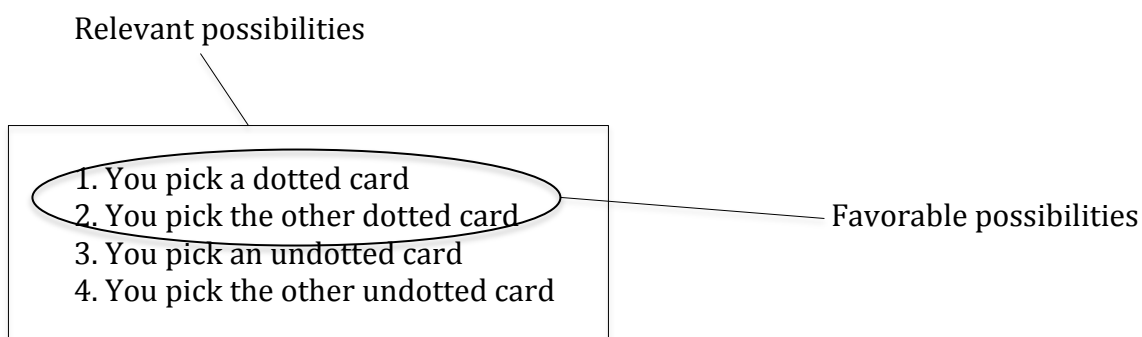
specific _____ class we associate with the event.

So let's go back to the example involving the four cards. Our question is: What's the probability that the card you pick will be dotted? Even though the answer to this question may be glaringly obvious, the example can be used to illustrate Insight 1. To work out the answer to the question, we can first note the event we're dealing with, and then identify both the relevant possibilities and the favorable possibilities:

Event: You pick a dotted card

General class of events: You pick one of the four cards

More specific class of events: You pick one of the two dotted cards



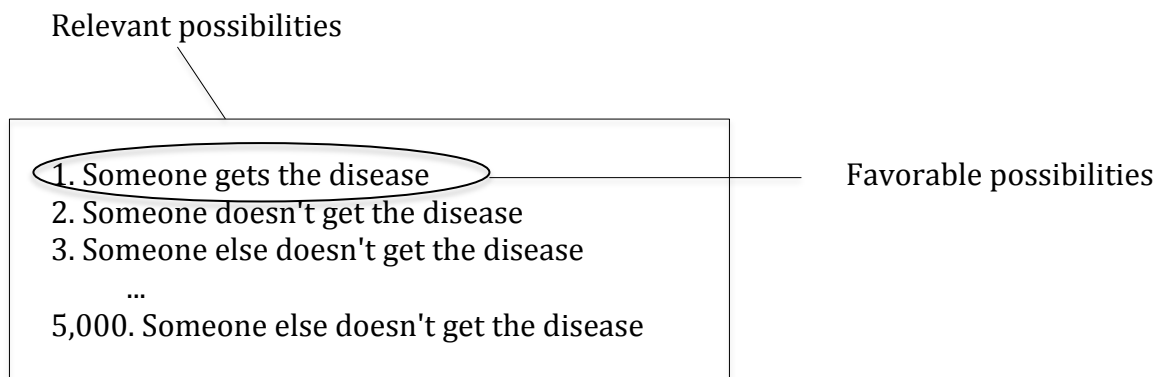
Since there are four relevant possibilities here, and two of them are favorable, Insight 1 tells us that the probability of the event in question is $2/4$ ($= 1/2 = 50\%$).

Now consider the example involving the low incidence of a disease. To answer the question, "What's the probability that someone gets the disease?" we can apply Insight 1 as follows:

Event: Someone gets the disease

General class of events: Someone is in a group of 5,000 people

More specific class of events: Someone is in a group of 5,000 people and gets the disease



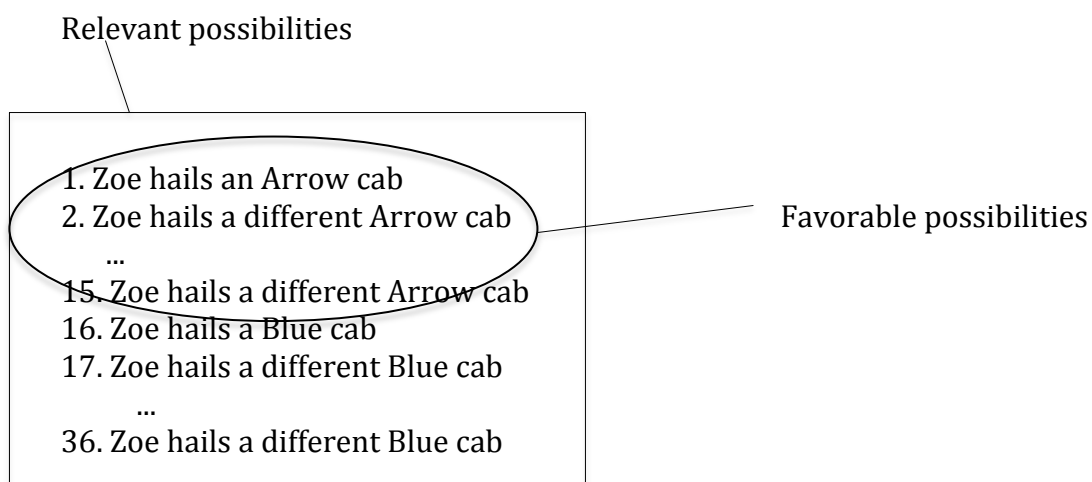
Thus, following Insight 1, we can say that the probability of the event in question is $1/5,000$ (=0.0002%).

Here's another, more complex example that illustrates Insight 1. Suppose that there are only two taxicab companies in a town: Arrow and Blue. Arrow has 15 cabs on the road and Blue has 21. One night, Zoe hails a cab at random in the town. What's the probability that Zoe hails an Arrow cab?

Event: Zoe hails an Arrow cab

General class of events: Zoe hails one of the town's cabs

More specific class of events: Zoe hails one of the town's Arrow cabs



Since there are here 15 favorable possibilities out of 36 relevant possibilities, Insight 1 tells us that the probability that the probability of our event $15/36$ (= $5/12$ = approx. 42%).

When estimating the probability of an event, it's important to think as clearly as you can about the relevant possibilities. Very often, significant mistakes in estimating the probability of an event can be traced back to confusion or unclear thinking about what the relevant possibilities are.

To illustrate this point, consider **the Monty Hall Problem**. Monty Hall was the host of the television game show, "Let's Make a Deal." The problem comes from a description of a particular game that was played on the show: A contestant is shown three closed doors, and told that behind just one of them there's a great prize; the other two are loser doors, behind which there's no prize worth mentioning. The contestant then picks one of the doors. But, before the door she picks is opened, Monty opens one of the other doors, revealing that it's a loser. (Monty, of course, knows what's behind each door, and would never have opened anything but a loser door.) Monty then asks the contestant whether she would like to switch her original pick to the other unopened door.

Should the contestant switch? If you're like most people when first presented with this problem, you'll be inclined to say that there's no particular reason for her to switch: her chances of winning by switching are just the same as her chances of winning by staying with her original pick.

Interestingly enough, however, this is incorrect. In fact, the contestant should switch, because she *doubles* her chances of winning by switching.

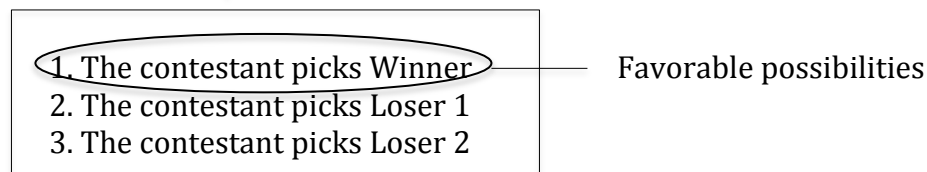
The reason so many people are inclined to get this wrong upon first consideration has to do with a failure to take into account the relevant possibilities of the situation. To see this, first consider the probability of the contestant's winning with her initial pick, before Monty opens his door. We can easily apply Insight 1 to answer this question as follows:

Event: The contestant picks a winning door

General class of events: The contestant picks one of the doors

More specific class of events: The contestant picks one of the winning doors

Relevant possibilities



Thus, applying Insight 1, the probability of the contestant picking the winning door with her initial pick is $1/3$.

Now consider the different question of the probability that the contestant picks the winning door by switching from her initial pick, after Monty has opened his losing door. In asking this, we're essentially asking: What's the probability of the contestant switching to the winning door? To use Insight 1 to get an answer, we can first identify the event and its associated general class and more specific class of events:

Event: The contestant switches to a winning door

General class of events: The contestant switches to one of the doors

More specific class of events: The contestant switches to one of the winning doors

You might then be inclined to think that the relevant possibilities here are as follows:

1. The contestant switches to Winner from Loser 1
2. The contestant switches to Winner from Loser 2

3. The contestant switches to Loser 1 from Winner
4. The contestant switches to Loser 1 from Loser 2
5. The contestant switches to Loser 2 from Winner
6. The contestant switches to Loser 2 from Loser 1

Those are all the ways switching to another door (i.e., members of the general class of events) after Monty has opened his door, right?

Well, no. In fact, there are *too many* relevant possibilities listed here. After Monty opens his door, we know (a) that the contestant can't switch from one losing door to another (because, since Monty has revealed a losing door, such a switch would mean that all three of the doors are losing doors). We also know (b) that the contestant can't switch to *both* losing doors from the winning door, only to one or the other of them (because by revealing one of the losing doors, Monty has in effect given the contestant that losing door, and she can't switch to something she already has). (Notice by contrast that, for all we know from Monty's revelation, it is still possible for the contestant to switch to the winning door from both losing doors; this is because it is possible to switch *from* something you already have.)

In the light of these two things we know, then, we can rule out some of the above suggested relevant possibilities to get an accurate list: Both (4) and (6) have to go in the light of (a). And either (3) or (5) has to go in the light of (b); let's just eliminate (5):

1. The contestant switches to Winner from Loser 1
2. The contestant switches to Winner from Loser 2
3. The contestant switches to Loser 1 from Winner
- ~~4. The contestant switches to Loser 1 from Loser 2~~
- ~~5. The contestant switches to Loser 2 from Winner~~
- ~~6. The contestant switches to Loser 2 from Loser 1~~

Now that we have an accurate view of the relevant possibilities for our probability question, we have simply to note the proportion of these that are favorable, and apply Insight 1:

Event: The contestant switches to a winning door

General class of events: The contestant switches to one of the doors

More specific class of events: The contestant switches to one of the winning doors

Relevant possibilities

1. The contestant switches to Winner from Loser 1
 - ~~2. The contestant switches to Winner from Loser 2~~
 3. The contestant switches to Loser 1 from Winner

Favorable possibilities

Thus, the probability that the contestant switches to the winning door (after Monty reveals his losing door) is $2/3$. Since this probability is twice that of the contestant initially picking the winning door (which is the probability of her winning by sticking with her initial choice), it seems that, indeed, she should switch after all.

Exercise 9.1: For each of the following examples, identify the event, the relevant possibilities, and the favorable possibilities; then calculate the probability of the event in accordance with Insight 1.

There are 52 cards in a standard deck of playing cards. 13 of these are spades. You're presented with a randomly shuffled deck, and asked to draw a single card. What's the probability that you draw a spade?

200 students took an exam. 100 of the students got a C or lower on the exam, 72 got a B, and the rest got an A. What's the probability that a student got an A on the exam?

You know the next-door neighbors have two young children. What's the probability that both of the children are girls?

You know the next-door neighbors have two young children. You've just learned, moreover, that one of the children is a girl. Now, what's the probability that both of the children are girls?

Exercise 9.1 Solutions

There are 52 cards in a standard deck of playing cards. 13 of these are spades. You're presented with a randomly shuffled deck, and asked to draw a single card. What's the probability that you draw a spade?

Event: You draw a spade

General class of events: You draw one of the cards

More specific class of events: You draw one of the spade cards

Relevant possibilities

1. You draw a spade
2. You draw a different spade
- ...
13. You draw a different spade
14. You draw a non-spade
15. You draw a different non-spade
- ...
52. You draw a different non-spade.

Favorable possibilities

Since there are 13 favorable possibilities of 52 relevant possibilities, Insight 1 tells us that the probability of the event in question is $13/52 (= 1/4 = 25\%)$.

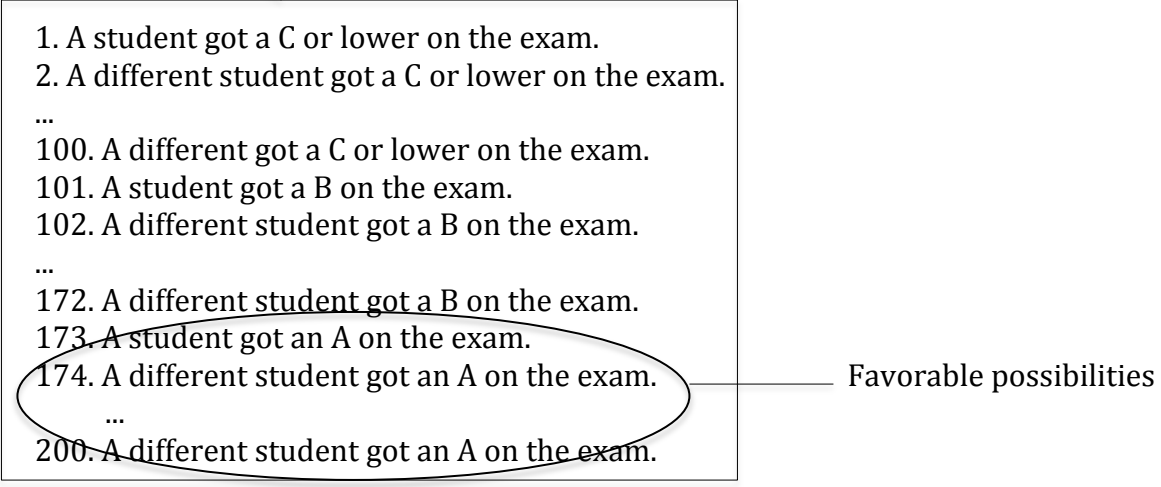
200 students took an exam. 100 of the students got a C or lower on the exam, 72 got a B, and the rest got an A. What's the probability that a student got an A on the exam?

Event: A student got an A on the exam

General class of events: A student got one of the grades on the exam

More specific class of events: A student got one of the A grades on the exam

Relevant possibilities

- 
1. A student got a C or lower on the exam.
 2. A different student got a C or lower on the exam.
 - ...
 100. A different student got a C or lower on the exam.
 101. A student got a B on the exam.
 102. A different student got a B on the exam.
 - ...
 172. A different student got a B on the exam.
 173. A student got an A on the exam.
 174. A different student got an A on the exam.
 - ...
 200. A different student got an A on the exam.

Favorable possibilities

There are 200 relevant possibilities, and 28 favorable possibilities. So, according to Insight 1, the probability of the event in question is $28/200 (= 14\%)$.

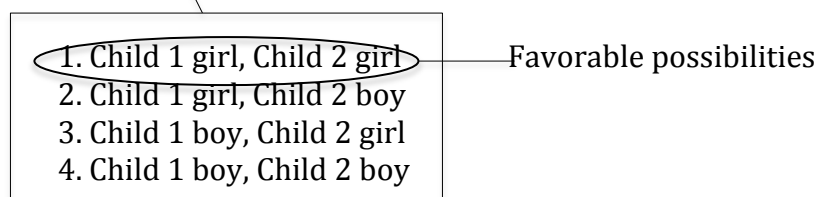
You know the next-door neighbors have two young children. What's the probability that both of the children are girls?

Event: The neighbors have two girls.

General class of events: The neighbors have two children.

More specific class of events: The neighbors have two children both of which are girls.

Relevant possibilities



Of the four relevant possibilities, one is favorable. So the probability that the neighbors have two girls is $1/4$.

You know the next-door neighbors have two young children. You've just learned, moreover, that one of the children is a girl. Now, what's the probability that both of the children are girls?

Event: The neighbors have two girls.

General class of events: The neighbors have two children at least one of which is a girl.

More specific class of events: The neighbors have two children both of which are girls.

Relevant possibilities



Of the three relevant possibilities, one is favorable. So, now, the probability that both children are girls is $1/3$.

Our next two insights concern the probabilities of complex events, i.e., events that include other events as constituents. Insight 2 concerns the probabilities of joint events, i.e. events of the "both-and" sort:

Insight 2: Joint events. The probability of two events together is _____

_____ the probability of the first multiplied by the probability of the second given the first.

I.e.,

$$P(A \text{ and } B) = P(A) \times P(B \text{ given } A).$$

To talk about the probability of one event *given* another is to talk about the probability of one event occurring *on the assumption that the other has occurred*. For example, suppose that the table in front of you has a bowl with three dark chocolates and two white chocolates in it. You reach in twice and grab a chocolate without replacing it. If we ask "What's the probability of your grabbing a dark chocolate the second time given that you grabbed a dark chocolate the first time," we're asking about grabbing a dark chocolate the second time on the assumption that you grabbed a dark chocolate the first time. This sort of "given that" probability is known as "**conditional probability**"--the probability of one event conditional upon another.

Consider now how to apply Insight 2 to the example just mentioned: What's the probability that you get a dark chocolate both times? According to Insight 2:

$$\begin{aligned}
 P(\text{Dark first time and Dark second time}) &= P(\text{Dark first time}) \times P(\text{Dark second time} \\
 &\text{given Dark first time}) \\
 &= 3/5 \times 2/4 \\
 &= 6/20 \\
 &= 3/10 \\
 &= 30\%
 \end{aligned}$$

In many situations, two events are **independent** of each other, i.e., the occurrence of the one has no bearing likelihood of the other occurring. In such cases, the probability of the second given the first turns out to be just the same as the probability of the second alone. Thus, in such cases, Insight 2 happily reduces to the simpler

$$P(A \text{ and } B) = P(A) \times P(B).$$

This can be seen by changing the previous example this way: you replace the chocolate both times after grabbing it (perhaps because both times you decide you don't want it). The events of getting dark the first time and getting dark the second time are now independent of each other, and applying Insight 2 amounts to applying the simpler formula above:

$$\begin{aligned}
 P(\text{Dark first time and Dark second time}) &= P(\text{Dark first time}) \times P(\text{Dark second time}) \\
 &= 3/5 \times 3/5 \\
 &= 9/25 \\
 &= 36\%
 \end{aligned}$$

Notice something that follows from Insight 2: **The probability of two events together is never higher than the probability of either event on its own.** (This, of course, assumes the obvious truth that the probability of an event can't exceed 100%.) **In fact, it's typically significantly lower.** In each of the examples above, for example, the calculated probability of the two events together is significantly lower than the probability of either event on its own.

This can be relevant to assessing the logical strength of arguments whose conclusions are about two events together. Thus, recall the first part of the argument we considered at the start of our discussion of probabilities:

- P1. There's a good chance that you'll meet your ex at the party tonight.
- P2. There's a good chance that you'll meet your boss at the party tonight.
- C1. There's a good chance both that you'll meet your ex at the party tonight and that you'll meet your boss at the party tonight.

Even construed as a nondeductive argument, this piece of reasoning isn't obviously logically strong. Suppose we interpret "good chance" throughout as "70% probability." Assuming that the relevant two events are independent, Insight 2 tells us that the probability of their both occurring together is still less than 50%.

Exercise 9.2: Use Insight 2 to answer the following questions.

Someone tosses a fair coin twice. What's the probability of her getting heads both times?

Suppose we estimate the probability that *The National Enquirer* will carry a lead report about a celebrity scandal in its next issue at 90%. And suppose that the probability that *The Globe and Mail* will carry such a report in its next issue given that *The National Enquirer* does is 2%. What's the probability that both *The National Enquirer* and *The Globe and Mail* will carry a lead report about a celebrity scandal in their next issue?

Exercise 9.2 Solutions

Someone tosses a fair coin twice. What's the probability of her getting heads both times?

Since the two tosses are independent events, Insight 2 tells us that we can answer this question by simply multiplying their separate probabilities:

$$\begin{aligned}
 P(\text{Heads first toss and Heads second toss}) &= P(\text{Heads first toss}) \times P(\text{Heads second toss}) \\
 &= 1/2 \times 1/2 \\
 &= 1/4 \\
 &= 25\%
 \end{aligned}$$

Suppose we estimate the probability that *The National Enquirer* will carry a lead report about a celebrity scandal in its next issue at 90%. And suppose that the probability that *The Globe and Mail* will carry such a report in its next issue given that *The National Enquirer* does is 2%. What's the probability that both *The National Enquirer* and *The Globe and Mail* will carry a lead report about a celebrity scandal in their next issue?

Insight 2 makes this calculation easy:

$$\begin{aligned}
 P(\text{Enquirer reports and Globe reports}) &= P(\text{Enquirer reports}) \times P(\text{Globe reports given Enquirer reports}) \\
 &= 90\% \times 2\% \\
 &= 1.8\%
 \end{aligned}$$

Our third insight from the probability calculus concerns alternative events, i.e., complex events of the "either-or" sort:

Insight 3: Alternative events. The probability of either one event or another is _____

_____ the probability is the first, added the probability of the second minus the probability of them both together.

I.e.,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Insight 3 can be useful in many situations because in many situations we might be very interested in knowing the chances that either one event or another occurs, but don't much care which of the two occurs. For example, suppose there is an election for the presidency of a society coming up, and we think it would be a very good thing if either of two candidates, Candidate 1 or Candidate 2, is elected, but we don't much care which of the two is elected. From the polls, we know that there's about a 20% chance of Candidate 1 being elected, and about a 40% chance that Candidate 2 will be elected. Should we be optimistic that either Candidate 1 or Candidate 2 will be elected?

We know that both can't be elected, because there can be only one president. So we know that the probability of Candidate 1 being elected and Candidate 2 being elected is 0%. (When two events cannot occur together, i.e. the probability of both events together is zero percent, we say that the two events are **mutually exclusive.**) And we can thus easily apply Insight 3 to get an answer to our question:

$$\begin{aligned}
 P(\text{Candidate 1 elected or Candidate 2 elected}) &= P(\text{Candidate 1 elected}) + P(\text{Candidate 2 elected}) - P(\text{Candidate 1 elected and Candidate 2 elected}) \\
 &= 20\% + 40\% - 0\% \\
 &= 60\% - 0\% \\
 &= 60\%
 \end{aligned}$$

So, it seems, we should indeed be at least somewhat optimistic: There's about a 60% chance that either Candidate 1 will be elected or Candidate 2 will be elected.

(A happy side note: You'll never have to worry about applying Insight 3 to examples in which the two individual events are mutually exclusive but in which their separate probabilities add up to more than 100%. This is because no events, including joint ones, can have a probability that exceeds 100%, as we have seen; hence examples of this sort won't ever be found.)

A second example of how Insight 3 may be applied: You really enjoy two current television shows: AMC's *The Walking Dead* and HBO's *True Detective*. You know (suppose) that there's about a 50% chance of these two shows both airing tonight. But you can't remember for sure whether either one of them airs tonight. Based on the limited evidence of your memory, however, you think there's a 80% chance that *True Detective* airs tonight, and a 10% chance that *Walking Dead* airs tonight. What, following Insight 3, should you believe to be the probability that either one or the other show airs tonight?

$$\begin{aligned}
 P(\text{Walking Dead airs or True Detective airs}) &= P(\text{Walking Dead airs}) + P(\text{True Detective airs}) - P(\text{Walking Dead airs and True Detective airs}) \\
 &= 80\% + 10\% - 50\% \\
 &= 90\% - 50\% \\
 &= 40\%
 \end{aligned}$$

So you should believe that there's about a 40% probability that at least one of the two shows will air tonight.

Exercise 9.3: Use Insight 3 to answer the following questions.

A crime was committed. Among the usual suspects are Shifty and Shady. The probability that Shifty and Shady both committed the crime together is pretty low: about 10%. However, the probability that Shifty committed the crime on his own is about 60%, and the probability that Shady committed the crime on his own around 30%. What's the probability that either Shifty or Shady committed the crime?

Two sisters, Jiejie and Meimei, are both taking a philosophy course. They never both ask a question in class together. However, 5 times out of 10, Meimei asks a question in class. By contrast, Jiejie asks a question in class only about 20% of the time. What's the probability the either Jiejie or Meimei will ask a question in class?

Exercise 9.3 Solutions

A crime was committed. Among the usual suspects are Shifty and Shady. The probability

that Shifty and Shady both committed the crime together is pretty low: about 10%. However, the probability that Shifty committed the crime on his own is about 60%, and the probability that Shady committed the crime on his own around 30%. What's the probability that either Shifty or Shady committed the crime?

According to Insight 3:

$$\begin{aligned}
 P(\text{Shifty did it or Shady did it}) &= P(\text{Shifty did it}) + P(\text{Shady did it}) - P(\text{Shifty did it and Shady did it}) \\
 &= 60\% + 30\% - 10\% \\
 &= 90\% - 10\% \\
 &= 80\%
 \end{aligned}$$

Two sisters, Jiejie and Meimei, are both taking a philosophy course. They never both ask a question in class together. However, 5 times out of 10, Meimei asks a question in class. By contrast, Jiejie asks a question in class only about 20% of the time. What's the probability the either Jiejie or Meimei will ask a question in class?

We can use Insight 3 to answer as follows:

$$\begin{aligned}
 P(\text{Jiejie asks or Meimei asks}) &= P(\text{Jiejie asks}) + P(\text{Meimei asks}) - P(\text{Jiejie asks and Meimei asks}) \\
 &= 20\% + 50\% - 0\% \\
 &= 70\% - 0\% \\
 &= 70\%
 \end{aligned}$$