Techniques of Integration

For MAT 1320, Fall 2011

I. The Substitution Rule (§ 4.5, 5.7)

Suppose we want to find an indefinite integral $\int f(x)dx$. First try to see if we can re-write the integrand so that we can use the formulas directly. If not, try this method of variable substitution, before using the other rules.

Step 1. Find an intermediate variable $u = g(x)$.

Rules of thumb: (i) When use u instead of x, the integrand become simpler.

(ii) The derivative $g'(x)$ is a factor of the integrand.

Step 2. Divide the integrand by the derivative of $g(x)$, and write du instead of dx:

$\int f(x)dx = \int f(x)\frac{1}{g'(x)}du$. (Since $\frac{du}{dx} = g'(x)$, $dx = \frac{1}{g'(x)}du$).

Step 3. Rewrite the integrand as a function of u:

$$\frac{f(x)}{g'(x)} = h(u).$$

Note that, after this step, no x is in the integrand.

Step 4. Find $\int h(u)du = H(u) + C$.

Note that, this integrand should be easy to find. If not, either you used a wrong substitution, or this method fails.

Step 5. The final result is $H(g(x)) + C$.

Examples

(i) $\int \sin(3x)dx$, $u = 3x$, $u' = 3$.

$$\int \sin(3x)dx = \int \sin(3x)\frac{1}{3}du = \frac{1}{3}\int \sin u\,du = \frac{1}{3}(-\cos u) + C = -\frac{1}{3}\cos(3x) + C.$$

(ii) $\int e^{-3t}dt$, $u = -3t$, $u' = -3$.

$$\int e^{-3t}dt = \int e^{-3t}\frac{1}{-3}du = -\frac{1}{3}\int e^{u}du = -\frac{1}{3}e^{-3t} + C.$$

(iii) $\int \sqrt{1+2x}dt$, $u = 1 + 2x$, $u' = 2$.

$$\int \sqrt{1+2x}dt = \int \sqrt{1+2x}\frac{1}{2}du = \frac{1}{2}\int \sqrt{u}du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{1}{3}(1+2x)^{3/2} + C.$$

(iv) $\int te^{t^2}dt$, $u = t^2$, $u' = 2t$.

$$\int te^{t^2}dt = \int te^{t^2}\frac{1}{2t}du = \frac{1}{2}\int e^{t^2}du = \frac{1}{2}\int e^{u}du = \frac{1}{2}e^{u} + C = \frac{1}{2}e^{t^2} + C.$$

(v) $\int \frac{1}{x\ln x}dx$, $u = \ln x$, $u' = 1 / x$.

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{x\ln x}xdu = \int \frac{1}{\ln x}du = \int \frac{1}{u}du = \ln u + C = \ln(\ln x) + C.$$

(vi) $\int \frac{1}{\sqrt{9-x^2}}dx$, $u = x / 3$, $u' = 1 / 3$.

$$\int \frac{1}{\sqrt{9-x^2}}dx = \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}dx = \frac{1}{3}\int \frac{3}{\sqrt{1-\left(\frac{x}{3}\right)^2}}du == \int \frac{1}{\sqrt{1-u^2}}du = \arcsin u + C$$

$$= \arcsin\left(\frac{x}{3}\right) + C.$$

(vii) $\int \frac{x+1}{x^2+2x+5}dx$, $u = x^2 + 2x + 5$, $u' = 2(x + 1)$.

$$\int \frac{x+1}{x^2+2x+5}dx = \int \frac{x+1}{x^2+2x+5}\left(\frac{1}{2(x+1)}\right)du = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+2x+5| + C.$$

(Absolute value sign can be omitted).

(viii) $\int \frac{1}{x^2+2x+5}dx$. $x^2 + 2x + 5 = (x + 1)^2 + 4$. $u = \frac{x+1}{2}$, $u' = \frac{1}{2}$.

$$\int \frac{1}{x^2+2x+5}dx = \int \frac{2}{(x+1)^2+4}du = \frac{1}{2}\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}du = \frac{1}{2}\int \frac{1}{u^2+1}du = \frac{1}{2}\arctan u + C$$

$$= \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$

(ix) $\int \frac{3x-5}{x^2+2x+5}dx$. $\frac{3x-5}{x^2+2x+5} = \frac{3(x+1)-8}{x^2+2x+5}$.

$$\int \frac{3x-5}{x^2+2x+5}dx = 3\int \frac{x+1}{x^2+2x+5}dx - 8\int \frac{1}{x^2+2x+5}dx = \frac{3}{2}\ln(x^2+2x+5) - 4\arctan\left(\frac{x+1}{2}\right) + C.$$

(x) $\int \frac{e^{2x}}{1+e^x}dx$. $u = 1 + e^x$, $u' = e^x$, $e^x = u - 1$.

$$\int \frac{e^{2x}}{1+e^x}dx = \int \frac{e^{2x}}{1+e^x}\frac{1}{e^x}du = \int \frac{e^x}{1+e^x}du = \int \frac{u-1}{u}du = \int du - \int \frac{1}{u}du = u - \ln u + C$$
$$= e^x - \ln(1+e^x) + C'. \quad C' = 1 + C.$$

(xi) $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$. $u = e^x - e^{-x}$, $u' = e^x + e^{-x}$.

$$\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx = \int \frac{1}{e^x-e^{-x}}du = \ln|e^x - e^{-x}| + C.$$

(xii) $\int \frac{x}{1+\sqrt{x}}dx$. $u = 1 + \sqrt{x}$, $u' = \frac{1}{2\sqrt{x}}$.

$$\int \frac{x}{1+\sqrt{x}}dx = \int \frac{x}{1+\sqrt{x}}(2\sqrt{x})du = 2\int \frac{x^{3/2}}{1+\sqrt{x}}du = 2\int \frac{(u-1)^3}{u}du = 2\int \frac{u^3-3u^2+3u-1}{u}du$$
$$= 2\left(\int u^2du - 3\int udu + 3\int du - \int \frac{1}{u}du\right) = \frac{2}{3}u^3 - 3u^2 + 6u - 2\ln u + C$$
$$= \frac{2}{3}(1+\sqrt{x})^3 - 3(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + C.$$

$$= \frac{2}{3}(1 + 3\sqrt{x} + 3x + x^{3/2}) - 3(1 + 2\sqrt{x} + x) + 6(1 + \sqrt{x}) - 2\ln(1 + \sqrt{x}) + C$$

$$= 2\sqrt{x} - x + \frac{2}{3}x\sqrt{x} - 2\ln(1 + \sqrt{x}) + C'.$$

(xiii) $\int \cos^2 x dx$. $\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$.

$\int \cos^2 x dx = \frac{1}{2}\int (1 + \cos(2x))dx = \frac{1}{2}\left(\int dx + \int \cos(2x)dx\right)$. $u = 2x$, $u' = 2$.

$$\frac{1}{2}\left(\int dx + \int \cos(2x)dx\right) = \frac{1}{2}\left(x + \frac{1}{2}\int \cos u du\right) = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$$

(xiv) $\int \tan x dx$. $u = \cos x$, $u' = -\sin x$.

$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{\sin x}{\cos x}\frac{1}{\sin x}du = -\int \frac{1}{u}du = -\ln|\cos x| + C.$$

(xv) $\int \sin^2 x \cos^3 x dx$. $u = \sin x$, $u' = \cos x$.

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cos^3 x \frac{1}{\cos x}du = \int \sin^2 x \cos^2 x du = \int u^2(1 - u^2)du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$
$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$$

Using variable substitution to find definite integrals:

$$\int_a^b f(x)dx = \int_{g(a)}^{g(b)} \frac{f(x)}{g'(x)}du = \int_{g(a)}^{g(b)} h(u)du = H(g(b)) - H(g(a)).$$

Examples

(i) $\int_0^3 \frac{x}{\sqrt{x+1}+3}dx = \int_4^5 \frac{x}{\sqrt{x+1}+3}(2\sqrt{x+1})du = 2\int_4^5 \frac{x\sqrt{x+1}}{\sqrt{x+1}+3}du$,

Since $u = \sqrt{x+1} + 3$, $x + 1 = (u-3)^2$, $x = (u-3)^2 - 1 = u^2 - 6u + 8$.

$$2\int_4^5 \frac{x\sqrt{x+1}}{\sqrt{x+1}+3}du = 2\int_4^5 \frac{(u^2 - 6u + 8)(u-3)}{u}du = 2\int_4^5 \frac{u^3 - 9u^2 + 26u - 24}{u}du$$

$$= 2\int_4^5 \left(u^2 - 9u + 26 - \frac{24}{u}\right) du = 2\left[\frac{u^3}{3} - \frac{9u^2}{2} + 26u - 24\ln|u|\right]_{u=4}^5$$
$$= \frac{35}{3} - 48\ln\frac{5}{4}.$$

(ii) $\int_0^{\pi/2} \cos^3 x dx$. $u = \sin x$, $u' = \cos x$. $u(0) = 0$, $u(\pi/2) = 1$.

$$\int_0^{\pi/2} \cos^3 x dx = \int_0^1 \cos^3 x \frac{1}{\cos x} du = \int_0^1 \cos^2 x du = \int_0^1 (1-u^2) du = \left[u - \frac{1}{3}u^3\right]_{u=0}^1 = \frac{2}{3}.$$

(iii) $\int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^4}} dx$. Let $u = x^2$. $u' = 2x$.

$$\int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-x^4}} du = \frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}} du = \frac{1}{2}[\arcsin u]_{u=0}^{1/2} = \frac{\pi}{12}.$$

II. Integration by Parts (§ 5.6)

Integration by parts are used to find the integral of some function that are product of two functions: $\int F(x)g(x)dx$. The idea is to integrate one factor first, then the other.

Step 1. Choose $g(x)$ to integrate first with rules of thumb: (a) Function $g(x)$ is easy to integrate, and (b) $F(x)$ has a simple derivative.

Let $G(x)$ be an antiderivative of $g(x)$. Then write

$$\int F(x)g(x)dx = \int F(x)dG(x).$$

Step 2. Use the formula $\int F(x)dG(x) = F(x)G(x) - \int G(x)dF(x)$.

Step 3. Write the differential $dF(x) = F'(x)dx$. Let $f(x) = F'(x)$, we have

$$\int F(x)dG(x) = F(x)G(x) - \int G(x)dF(x) = F(x)G(x) - \int G(x)f(x)dx.$$

The last integral $G(x)f(x)$ should be easier than the original integral.

To make the formula easier to memorize, denote $F(x)$ by u and denote $G(x)$ by v. Then the formula becomes

$\int udv = uv - \int vdu$.

Don't abuse this method: (a) Try it only if the method of variable substitution failed. (b) Use this method only to problems similar to examples that you have seen.

Examples

a. Basic examples

(i) $\int xe^x dx = e^x(x - 1) + C,$

(ii) $\int x\cos x dx = \cos x + x\sin x + C$.

(iii) $\int \sqrt{x}\ln x dx = \int \ln x d\left(\frac{2}{3}x^{3/2}\right) = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$

(vi) $\int \frac{x}{\cos^2 x}dx = \int xd\tan x = x\tan x - \int \tan xdx = x\tan x + \ln|\cos x| + C.$

b. Use this method twice:

(v) $\int x^2e^x dx = e^x(x^2 - 2x + 2) + C,$

(vi) $\int x^2\sin xdx = -x^2\cos x + 2\int x\cos xdx = -x^2\cos x + 2\left(x\sin x - \int \sin xdx\right)$
$= -x^2 \cos x + 2x \sin x + 2 \cos x + C.$

c. Use integration by parts to functions with only one factor mainly in the cases where the integrand has a simple derivative:

(vii) $\int \ln xdx. = x\ln x - x + C.$

(viii) $\int \arcsin xdx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx = x\arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{u}}du$

$= x\arcsin x + \sqrt{1-x^2} + C$.

(ix) $\int (\ln x)^2 dx = x(\ln x)^2 - \int \frac{2x\ln x}{x}dx = x(\ln x)^2 - 2\int \ln xdx = x(\ln x)^2 - 2(x\ln x) + 2x + C.$

(x) $\int \ln(x + \sqrt{x^2+1})dx = x\ln(x + \sqrt{x^2+1}) - \int xd\left(\ln(x + \sqrt{x^2+1})\right)$

$$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}dx.$$

Use variable substitution $u = x^2$, $u' = 2x$.

$$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{2}\int \frac{1}{\sqrt{u}}du == x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + C.$$

d. Recursive:

(x)

$$\int e^x \sin x = \int \sin x de^x = e^x \sin x - \int e^x \cos x dx = e^x \sin x - \int \cos x de^x = e^x(\sin x - \cos x) - \int e^x \sin x dx.$$

$$\int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

(xi) $\int \cos^{2n} x dx = \int \cos^{2n-1} x d\sin x = \sin x \cos^{2n-1} x - \int \sin x d\cos^{2n-1} x$

$$= \sin x \cos^{2n-1} x + (2n-1)\int \sin^2 x \cos^{2n-2} x dx = \sin x \cos^{2n-1} x + (2n-1)\int (1-\cos^2 x)\cos^{2n-2} x dx$$

$$= \sin x \cos^{2n-1} x + (2n-1)\int \cos^{2n-2} dx - (2n-1)\int \cos^{2n} x dx.$$

$$2n\int \cos^{2n} x dx = \sin x \cos^{2n-1} x + (2n-1)\int \cos^{2n-2} x dx.$$

$$\int \cos^{2n} x dx = \frac{\sin x \cos^{2n-1} x}{2n} + \frac{2n-1}{2n}\int \cos^{2n-2} x dx.$$

When $n = 1$, $\int \cos^2 x dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}\int dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}x + C$.

When $n = 2$, $\int \cos^4 x dx = \frac{1}{4}\sin x \cos^3 x + \frac{3}{4}\int \cos^2 x dx = \frac{1}{4}\sin x \cos^3 x + \frac{3}{8}\sin x \cos x + \frac{3}{8}x + C$.

e. Combine integration by parts and variable substitution:

(xii) $\int \sin\sqrt{x}dx = 2\int u \sin u du = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$, $u = \sqrt{x}$.

(xiii) $\int x^3 \cos(x^2)dx = \frac{1}{2}\int u \cos u du = \frac{1}{2}(x^2 \sin(x^2) + \cos(x^2)) + C$.

(xiv) $\int e^{\sqrt{x}}dx = 2\int ue^u du = 2e^{\sqrt{x}}(\sqrt{x}-1)+C$;

Definite integral with integration by parts:

We may find the indefinite integral first, and then use the fundamental theorem of calculus, or, use the formula:

$$\int_a^b F(x)dG(x) = \left[F(x)G(x)\right]_{x=a}^b - \int_a^b G(x)dF(x) .$$

Examples

(xv) Evaluate $\int_1^4 \frac{\ln x}{\sqrt{x}}dx$.

Since $\int \frac{1}{\sqrt{x}}dx = \int x^{-1/2}dx = 2x^{1/2}+C$,

$$\int_1^4 \frac{\ln x}{\sqrt{x}}dx = \int_1^4 \ln x d(2x^{1/2}) = 2\left[x^{1/2}\ln x\right]_{x=1}^4 - 2\int_1^4 x^{-1/2}dx = 4\ln 4 - 2\left[\sqrt{x}\right]_{x=1}^4 = 8\ln - 4.$$

(xvi) $\int_0^1 x^2 \arctan x dx = \int_0^1 \arctan x d\left(\frac{x^3}{3}\right) = \frac{1}{3}[x^3\arctan x]_{x=0}^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}dx = \frac{\pi}{12} - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}dx$.

Use $u = 1 + x^2$ to evaluate the last integral:

$$\int_0^1 \frac{x^3}{1+x^2}dx = \frac{1}{2}\int_1^2 \frac{u-1}{u}du = \frac{1}{2}[u-\ln u]_{u=1}^2 = \frac{1}{2}(1-\ln 2) .$$

Finally, $\int_0^1 x^2\arctan x dx = \frac{\pi}{12} - \frac{1}{6}(1-\ln 2) = \frac{1}{12}(\pi - 2 + 2\ln 2)$.

III. Partial Fractions (App. G, § 5.7)

1. *Partial Fractions*

A *rational function* is $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. It is a *proper* rational function if the degree of $P(x)$ is strictly smaller than the degree of $Q(x)$; it is *improper*, otherwise. If $f(x)$ is an improper rational function, dividing $P(x)$ by $Q(x)$ by long division, we can write

$\frac{P(x)}{Q(x)} = p(x) + \frac{R(x)}{Q(x)}$, where $p(x)$ is a polynomial and $R(x)$ is a proper rational function.

The method of partial fraction is used to find the integral of proper rational functions.

Let $f(x) = \dfrac{P(x)}{Q(x)}$ be a proper rational function. Factorize $Q(x)$ into linear and irreducible quadratic factors. Then this function can be express as the sum of a number of *partial fractions* of the form $\dfrac{K}{(ax+b)^n}$ or $\dfrac{Mx+N}{(ax^2+bx+c)^n}$, where n is a positive integer, and K, M and N are constants, which can be identified by equality. In the second case, $b^2 - 4ac < 0$. By completing the square, using a variable substitution, we can always write it in the form

$$\frac{Mx+N}{(ax^2+bx+c)^n} = \frac{Au+B}{(u^2+1)^n}.$$

Example. $\dfrac{2x-3}{(9x^2-6x+5)^2} = \dfrac{2x-3}{((3x-1)^2+4)^2} = \dfrac{2x-3}{4^2\left(\left(\dfrac{3x-1}{2}\right)^2+1\right)^2}$. Let $u = \dfrac{3x-1}{2}$.

Then $x = \dfrac{1}{3}(2u+1)$.

$$\frac{2x-3}{(9x^2-6x+5)^2} = \frac{2x-3}{((3x-1)^2+4)^2} = \frac{2x-3}{4^2\left(\left(\frac{3x-1}{2}\right)^2+1\right)^2} = \frac{\frac{2}{3}(2u+1)-3}{16(u^2+1)^2} = \frac{\frac{1}{12}u-\frac{17}{24}}{(u^2+1)^2}.$$

2. *Standard Forms*

A linear factor of the form $(ax + b)^n$ in the denominator corresponds to a sum

$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \ldots + \dfrac{A_n}{(ax+b)^n}$, and a irreducible quadratic factor of the form $(ax^2 + bx + c)^n$ in the denominator corresponds to a sum

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

Example. Let $f(x) = \dfrac{x^5-2x^2+3}{x(x-5)^2(x^2-x+2)^2}$. Then the standard partial fraction is

$$\frac{x^5-2x^2+3}{x(2x-5)^2(x^2-x+2)^2} = \frac{A}{x} + \frac{B}{2x-5} + \frac{C}{(2x-5)^2} + \frac{Dx+E}{x^2-x+2} + \frac{Fx+G}{(x^2-x+2)^2}.$$

3. *Integrating Standard Forms*

(i) $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$,

(ii) $\int \frac{1}{(ax+b)^n}dx = -\frac{1}{a(n-1)(ax+b)^{n-1}} + C$, $n > 1$,

(iii) $\int \frac{u}{u^2+1}du = \frac{1}{2}\ln(u^2+1) + C$,

(iv) $\int \frac{u}{(u^2+1)^n}du = \frac{1}{2}\int v^{-n}dv = \frac{1}{2(1-n)}v^{1-n} + C = -\frac{1}{2(n-1)(x^2+1)^{n-1}} + C$, , where $v = u^2 + 1$, and $n > 1$.

(v) $\int \frac{1}{u^2+1}du = \arctan u + C$,

(vi) $\int \frac{1}{(u^2+1)^n}du$, $n > 1$. (The case will be covered later with trigonometric substitution).

4. *Examples*

Single roots:

(i) $\int \frac{2x^3-11x^2-2x+2}{2x^2+x-1}dx$.

$$\frac{2x^3-11x^2-2x+2}{2x^2+x-1} = x-6+\frac{5x-4}{(2x-1)(x+1)}.$$

$$\frac{5x-4}{(2x-1)(x+1)} = \frac{A}{2x-1}+\frac{B}{x+1} = \frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)}.$$

$5x - 4 = A(x + 1) + B(2x - 1)$.

Let $x = 1 / 2$. $-3 / 2 = (3 / 2)A$. $A = -1$. Let $x = -1$, $-9 = -3B$, $B = 3$.

$$\int \frac{2x^3-11x^2-2x+2}{2x^2+x-1}dx = \int (x-6)dx - \int \frac{1}{2x-1}dx + 3\int \frac{1}{x+1}dx$$

$$= \frac{x^2}{2} - 6x - \frac{1}{2}\ln|2x-1| + 3\ln|x+1| + C.$$

(ii) $\int \frac{2x^3 + x^2 - x + 10}{x(x-1)(x^2-2x+5)}\,dx$

$$\frac{2x^3 + x^2 - x + 10}{x(x-1)(x^2-2x+5)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2-2x+5}.$$

$2x^3 + x^2 - x + 10 = A(x-1)(x^2 - 2x + 5) + Bx(x^2 - 2x + 5) + (Cx + D)x(x - 1)$
$= (A + B + C)x^3 - (3A + 2B + C - D)x^2 + (7A + 5B - D)x - 5A.$

Compare both sides:

$A + B + C = 2,$
$3A + 2B + C - D = -1,$
$7A + 5B - D = -1,$
$-5A = 10.$

Hence, $A = -2$. Plug in this value to the other equations. We have

$B + C = 4.$ (1)
$2B + C - D = 5.$ (2)
$5B - D = 13.$ (3)

(2) – (1): $B - D = 1$. (4)
(3) – (4): $4B = 12$, $B = 3$.
$D = 2$, $C = 1$.

or

Let $x = 0$. $10 = -5A$, $A = -2$.
Let $x = 1$. $12 = 4B$, $B = 3$.
Let $x = -1$. $10 = 32 - 24 - 2C + 2D$. $-C + D = 1$.
Let $x = 2$. $28 = -10 + 30 + 4C + 2D$. $2C + D = 4$.

$C = 1$, $D = 2$.

$$\int \frac{2x^3 + x^2 - x + 10}{x(x-1)(x^2-2x+5)}\,dx = -2\int \frac{1}{x}\,dx + 3\int \frac{1}{x-1}\,dx + \int \frac{x+2}{x^2-2x+5}\,dx.$$

Completing the square and let $u = (x - 1) / 2$, $u' = 1 / 2$, and $x = 2u + 1$.

$$\int \frac{x+2}{x^2-2x+5}dx = \int \frac{x+2}{(x-1)^2+4}dx = \frac{1}{4}\int \frac{x+2}{\left(\frac{x-1}{2}\right)^2+1}dx = \frac{1}{2}\int \frac{2u+3}{u^2+1}du$$

$$= \frac{1}{2}\left(\int \frac{2u}{u^2+1}du + 3\int \frac{1}{u^2+1}du\right) = \frac{1}{2}\ln(u^2+1) + \frac{3}{2}\arctan u + C$$

$$= \frac{1}{2}\ln(x^2-2x+5) + \frac{3}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

Multiple roots:

(iii) $\int \frac{x}{(x+2)^2(x-1)}dx$.

$$\frac{x}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}.$$

$x = A(x + 2)(x - 1) + B(x - 1) + C(x + 2)^2$.

$x = 1$: $1 = 9C$. $C = 1 / 9$.
$x = -2$: $-2 = -3B$. $B = 2 / 3$.
$x = 0$: $0 = -2A + (2 / 3)(-1) + 4 / 9 = -2A - 2 / 9$. $A = -1 / 9$.

$$\int \frac{x}{(x+2)^2(x-1)}dx = -\frac{1}{9}\int \frac{1}{x+2}dx + \frac{2}{3}\int \frac{1}{(x+2)^2}dx + \frac{1}{9}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{9}\ln|x+2| - \frac{2}{3}\left(\frac{1}{x+2}\right) + \frac{1}{9}\ln|x-1| + C.$$

(iv) $\int \frac{x^3+x^2-5x-16}{(x+2)^2(x^2+1)}dx$. Let

$$\frac{x^3+x^2-5x-16}{(x+2)^2(x^2+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+1} = \frac{A(x+2)(x^2+1)+B(x^2+1)+(Cx+D)(x+2)^2}{(x+2)^2(x^2+1)}.$$

$A(x+2)(x^2+1)+B(x^2+1)+(Cx+D)(x+2)^2 = x^3 + x^2 - 5x - 16$.

Let $x = -2$. $5B = -8 + 4 + 10 - 16 = -10$. $B = -2$.
Let $x = 0$. $2A + B + 4D = -16$, $2A + 4D = -14$, $A + 2D = -7$. (1)
Let $x = 1$. $6A + 2B + 9C + 9D = -19$, $6A + 9C + 9D = -15$, $2A + 3C + 3D = -5$. (2)
Let $x = -1$. $2A + 2B - C + D = -11$. $2A - C + D = -7$. (3)

(2) + 3 × (3): $8A + 6D = -26$, $4A + 3D = -13$. (4)

(4) – 4 × (1): $-5D = 15$, $D = -3$.
From (1): $A = -7 - 2D = -1$,
From (3): $C = 7 + 2A + D = 2$.

Then $\dfrac{x^3 + x^2 - 5x - 16}{(x+2)^2(x^2+1)} = -\dfrac{1}{x+2} - \dfrac{2}{(x+2)^2} + \dfrac{2x-3}{x^2+1}$, and

$$\int \frac{x^3 + x^2 - 5x - 16}{(x+2)^2(x^2+1)}dx = -\int \frac{1}{x+2}dx - 2\int \frac{1}{(x+2)^2}dx + \int \frac{2x}{x^2+1}dx - 3\int \frac{1}{x^2+1}dx$$
$$= -\ln|x+2| + \frac{2}{x+2} + \ln(x^2+1) - 3\arctan x + C.$$

IV. Approximating Definite Integrals (§ 5.9)

Find an approximation of the definite integral $\int_a^b f(x)dx$.

Given n = number of subintervals.

$h = (b - a) / n$: length of subintervals.

$x_0 = a$, $x_1 = a + h$, … , $x_i = a + ih$, …, $x_n = a + nh = b$.

Left Rule:

$I \approx \mathrm{L}(n) = h(f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1}))$.

Right Rule:

$I \approx \mathrm{R}(n) = h(f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n))$.

If $f(x)$ is increasing on $[a, b]$, L(n) is an underestimate, and R(n) is an overestimate.

If $f(x)$ is decreasing on $[a, b]$, R(n) is an underestimate, and L(n) is an overestimate.

$|\mathrm{R}(n) - \mathrm{L}(n)| = h\,|f(b) - f(a)|$ is an error bound,

Trapezoidal Rule:

$I \approx \mathrm{T}(n) = \mathrm{LEFT}(n) + \mathrm{RIGHT}(n) = (h/2)[f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_{n-1}) + f(x_n)]$.

Midpoint rule:

$I \approx \mathrm{M}(n) = h(f(x_1^*) + f(x_2^*) + \dots + f(x_n^*))$, where x_i^* is the midpoint of x_{i-1} and x_i:

$x_i^* = (x_{i-1} + x_i) / 2 = a + (i - 1/2)h$.

If $f(x)$ is concave up on $[a, b]$, M(n) is an underestimate, and T(n) is an overestimate.

If $f(x)$ is concave down on $[a, b]$, T(n) is an underestimate, and M(n) is an overestimate.

Simpson's Rule:

$n = 2m$ is an even number. $h = (b - a) / n$.

$I \approx \mathrm{S}(n) = (1/3)(2\mathrm{M}(m) + \mathrm{T}(n))$
$= (h/3)[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + 2f(x_5) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$.

Examples

1. $\int_0^1 \sin x dx = [-\cos x]_{x=0}^1 = 1 - \cos 1 \approx 0.4596976\dots$, $n = 8$, $h = 0.125$.

i	*x*	sin(*x*)
0	0.000	0.000000
	0.125	0.124675
1	0.250	0.247404
	0.375	0.366273
2	0.500	0.479426
	0.625	0.585097
3	0.750	0.681639
	0.875	0.767544
4	1.000	0.841471
	L(4)	0.352117
	R(4)	0.562485
	T(4)	0.457301
	M(4)	0.460897
	S(4)	0.459708

Error of S(4) ≈ 0.00001.

2. $f(x) = x^2 / (1 + x)$. $a = 1$, $b = 4$, $n = 6$, $h = 0.25$.

$$\int_1^4 \frac{x^2}{x+1}\,dx \approx 5.4162907\ldots$$

i	x	$x^2/(x+1)$
0	1	0.500000
	1.25	0.694444
1	1.5	0.900000
	1.75	1.113636
2	2	1.333333
	2.25	1.557692
3	2.5	1.785714
	2.75	2.016667
4	3	2.250000
	3.25	2.485294
5	3.5	2.722222
	3.75	2.960526
6	4	3.200000
	L(6)	4.745635
	R(6)	6.095635
	T(6)	5.420635
	M(6)	5.414130
	S(6)	5.416402

Error of S(6) ≈ 0.00011.

V. Trigonometric Substitution (§ 5.7) (not covered in second midterm)

Trigonometric substitution is the second type of variable substitution. Instead of using an intermediate variable $u = g(x)$, let $x = g(u)$. Here function $g(x)$ must have an inverse. In most cases, $g(u)$ is a trigonometric function, called trigonometric substitution.

The problem is to find $\int f(x)dx$.

(i) If $f(x)$ contains $\sqrt{a^2 - x^2}$, $a > 0$, consider use $x = g(x) = a \sin u$, $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $\sqrt{a^2 - x^2} = a\sqrt{1-\sin^2 x} = a \cos x$.

(ii) If $f(x)$ contains $\sqrt{a^2 + x^2}$, $a > 0$, consider use $x = g(x) = a \tan u$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$. Then $\sqrt{a^2 + x^2} = a\sqrt{1+\tan^2 x} = \frac{a}{\cos x}$.

(iii) If $f(x)$ contains $\sqrt{x^2 - a^2}$, $a > 0$, consider use $x = \sec u$, $0 \le u < \pi/2$, or $\pi/2 < u \le \pi$. Then

$$\sqrt{x^2-1} = \sqrt{\sec^2 u - 1} = \sqrt{\tan^2 u} = \begin{cases} \tan u, & 0 \le u < \pi/2 \\ -\tan u, & \pi/2 < u \le \pi \end{cases}.$$

Step 1. Let $\int f(x)dx = \int f(x)g'(u)du$. Note that $dx/du = g'(u)$.

Step 2. Write $f(x)g'(u)$ as a function of u: $f(x)g'(u) = f(g(x))g'(x) = h(u)$.

$$\int f(x)dx = \int f(x)g'(u)du = \int h(u)du.$$

Step 3. Find an antiderivative $H(u)$ of $h(u)$.

$$\int f(x)dx = \int f(x)g'(u)du = \int h(u)du = H(u) + C.$$

Step 4. Use the inverse function $u = g^{-1}(x)$. The integral is $H(g^{-1}(x)) + C$.

In partial fraction we have one case left: We want to find the integral $\int \frac{1}{(u^2+1)^n}du$, $n > 1$.

Use variable substitution $u = \tan\theta$, $-\pi/2 < \theta < \pi/2$, $u' = \frac{1}{\cos^2\theta}$. Then $\frac{1}{(u^2+1)^n} = \frac{1}{\sec^{2n}\theta} = \cos^{2n}\theta$, and $\int \frac{1}{(u^2+1)^n}du = \int \cos^{2n}\theta \frac{1}{\cos^2\theta}d\theta = \int \cos^{2(n-1)}\theta d\theta$. The last integral can be solved recursively. The final solution is expressed in terms of $\sin\theta$ and $\cos\theta$. Since $\tan\theta = u$, $\sin\theta = \frac{u}{\sqrt{1+u^2}}$, and $\cos u = \frac{1}{\sqrt{1+u^2}}$.

Example. $\int \frac{4x}{(x-1)(x^2+1)^2}dx$.

$$\frac{4x}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1) = 4x$.

Let $x = 1$. $4A = 4$. $A = 1$.

$(A + B)x^4 - (B - C)x^3 + (2A + B - C + D)x^2 - (B - C + D - E)x + A - C - E = x$.

$B = -A = -1$, $C = B = -1$, $D = -2A - B + C = -2$, $B - C + D - E = -4$, $E = 4 + B - C + D = 2$.

$$\int \frac{x}{(x-1)(x^2+1)^2}dx = \int \frac{1}{x-1}dx - \int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx - \int \frac{2x}{(x^2+1)^2}dx + 2\int \frac{1}{(x^2+1)^2}dx .$$

$$\int \frac{1}{x-1}dx = \ln|x-1| + C ,$$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1) + C ,$$

$$\int \frac{1}{x^2+1}dx = \text{arctan } x + C,$$

$$\int \frac{2x}{(x^2+1)^2}dx = -\frac{1}{x^2+1} + C ,$$

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{1}{\sec^4 u}\sec^2 u du = \int \cos^2 u du = \frac{1}{2}\sin u \cos u + \frac{1}{2}u + C = \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x + C .$$

$$\int \frac{4x}{(x-1)(x^2+1)^2}dx = \ln|x-1| - \frac{1}{2}\ln(x^2+1) - \arctan x + \frac{1}{x^2+1} + \frac{x}{x^2+1} + \arctan x + C$$
$$= \ln\left|\frac{x-1}{\sqrt{x^2+1}}\right| + \frac{x+1}{x^2+1} + C .$$

Examples

(i) $\int_0^a \sqrt{a^2 - x^2}dx = a\int_0^a \sqrt{1-(x/a)^2}dx$. Let $x = a \sin u$, $-\pi/2 \le u \le \pi/2$.

$$a^2\int_0^{\pi/2}\cos^2 u dx = \frac{a^2}{2}\int_0^{\pi/2}(1+\cos(2u))du = \frac{a^2}{2}\left(u + \frac{\sin(2u)}{2}\right)_{u=0}^{\pi/2} = \frac{\pi a^2}{4} .$$

(ii) $\int \frac{1}{x^2\sqrt{x^2+4}}dx$.

Let $x = 2 \tan u$, $-\pi/2 < u < \pi/2$. $x' = 2 / \cos^2 u$. $\frac{1}{\sqrt{x^2+4}} = \frac{1}{\sqrt{4\tan^2 u + 4}} = \frac{1}{2\sqrt{\tan^2+1}} = \frac{\cos u}{2}$.

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \left(\frac{1}{4\tan^2 u}\right)\left(\frac{\cos u}{2}\right)\left(\frac{2}{\cos^2 u}\right)du = \frac{1}{4}\int \frac{\cos u}{\sin^2 u}du = \frac{1}{4}\left(-\frac{1}{\sin u}\right) + C$$

$= -\frac{1}{4\sin u} + C$. Since tan $u = x / 2$, sin u = $\frac{x/2}{\sqrt{1+(x/2)^2}} = \frac{x}{\sqrt{x^2+4}}$.

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = -\frac{\sqrt{x^2+4}}{4x} + C$$

(iii) $\int_0^{1/2} \frac{1}{(1-x^2)^{3/2}}dx$. Let $x = \sin u$, $-\pi/2 \le u \le \pi/2$. $(1-x^2)^{3/2} = \cos^3 u$.

$$\int_0^{1/2} \frac{1}{(1-x^2)^{3/2}}dx = \int_0^{\pi/6} \frac{1}{\cos^2 u}du = \left[\tan u\right]_{u=0}^{\pi/6} = \frac{\sqrt{3}}{3}.$$

(iv) $\int \frac{1}{x^2\sqrt{x^2-1}}dx$. Let $x = \sec u$, $0 \le u < \pi/2$, or $\pi/2 < u \le \pi$. $\sqrt{x^2-1} = |\tan u|$.

$$\int \frac{1}{x^2\sqrt{x^2-1}}dx = \int \frac{1}{\sec^2 u\,|\tan u|}\sec u \tan u\,du = \int \cos u \frac{\tan u}{|\tan u|}du.$$

When $0 \le u < \pi/2$, this integral is $\int \cos u\,du = \sin u + C$. In this interval, sin u, sec $u = x$ and cos $u = 1/x$ are positive. Hence, sin u = $\sqrt{1-\cos^2 u} = \sqrt{1-\frac{1}{x^2}} = \sqrt{\frac{x^2-1}{x^2}} = \frac{\sqrt{x^2-1}}{x}$.

When $\pi/2 < u \le \pi$, this integral is $\int (-\cos u)du = -\sin u + C$. In this interval, sin u is positive, sec $u = x$ and cos $u = 1/x$ are negative. Hence,

$$\sin u = \sqrt{1-\cos^2 u} = \sqrt{1-\frac{1}{x^2}} = \sqrt{\frac{x^2-1}{x^2}} = -\frac{\sqrt{x^2-1}}{x}.$$

Finally, in both cases, $\int \frac{1}{x^2\sqrt{x^2-1}}dx = \frac{\sqrt{x^2-1}}{x} + C$.

Note that sometimes ordinary variable substitution works, and you don't have to use trigonometric substitution. For example, integral $\int \frac{x}{\sqrt{1-x^2}}dx$ can be solved easily by variable substitution $u = 1 - x^2$.