Solutions Tutorial 6 MATH1104

1. $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$ $\quad |A-\lambda I| = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda)+18$

$= -20+4\lambda-5\lambda+18+\lambda^2$

$= \lambda^2-\lambda-2=0$

$(\lambda-2)(\lambda+1)=0$

$\lambda=2, \lambda=-1.$

If $\lambda=2$:

$$\begin{bmatrix} -6 & 6 & 0 \\ -3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1-x_2=0 \\ x_1=x_2 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If $\lambda=-1$: $\begin{bmatrix} -3 & 6 & 0 \\ -3 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$P=\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, D=\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, P^{-1}=-1\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$A^9 = PD^9P^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^9 & 0 \\ 0 & (-1)^9 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -512 & 1024 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -514 & 1026 \\ -513 & 1025 \end{bmatrix}$$

2. $\frac{3-4i}{2-3i} + \frac{5-4i}{2-2i} = \frac{(3-4i)(2+3i)}{13} + \frac{(5-4i)(2+2i)}{8}$

$= \frac{6+9i-8i+12}{13} + \frac{10+10i-8i+8}{8}$

$= \frac{18+i}{13} + \frac{18+2i}{8} = \frac{18}{13} + \frac{i}{13} + \frac{9}{4} + \frac{i}{4}$

$= \left(\frac{18}{13} + \frac{9}{4}\right) + \left(\frac{1}{13} + \frac{1}{4}\right)i = \left(\frac{72+117}{52}\right) + \left(\frac{4+13}{52}\right)i$

$= \boxed{\frac{189}{52} + \frac{17}{52}i}$

3. $\begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+2 = 3-4\lambda+\lambda^2+2$

$= \lambda^2 - 4\lambda + 5 = 0$

$=$

$\lambda = \frac{4 \pm \sqrt{16-4(5)}}{2} = \frac{4 \pm 2i}{2}$

$= 2 \pm i$

$\underline{\lambda = 2+i:}$

$\begin{bmatrix} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1-i & 0 \\ -1-i & -2 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 + (1+i)R_1} \begin{bmatrix} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (-1+i)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1+i \\ 1 \end{bmatrix}.$

$\underline{\lambda = 2-i:}$ $\begin{bmatrix} \overline{-1+i} \\ 1 \end{bmatrix} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ is a basis for $2-i$.

4. $\begin{vmatrix} 3-\lambda & 1 \\ -2 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda)+2$

$= 15-8\lambda+\lambda^2+2 = \lambda^2-8\lambda+17=0$

$\lambda = \dfrac{8 \pm \sqrt{64-4(17)}}{2} = \dfrac{8\pm 2i}{2} = 4 \pm i$

$\lambda = 4+i$: $\begin{bmatrix} -1-i & 1 & 0 \\ -2 & 1-i & 0 \end{bmatrix} \xrightarrow{R_2=R_2+(-1+i)R_1} \begin{bmatrix} -1-i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(-1-i)x_1 + x_2 = 0$

$(-1-i)x_1 = -x_2$

$x_1 = \dfrac{-1}{-1-i}x_2$

$= \dfrac{(-1)(-1+i)}{2}x_2$

$= \left(\dfrac{1-i}{2}\right)x_2$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}$

so $\begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$)

$\lambda = 4-i$: The conjugate of previous vector is the basis for $4-i$.

so $\begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$).

5. a) $u \cdot v = 12+4-6+1 = ~~11~~$ $= 14$.

b) $u \cdot v = 0 + ~~6~~ + 6 - 2 + 4 + 0$

6. $\dfrac{u}{\|u\|} = \dfrac{(-1,2,5,-2)}{\sqrt{1+4+25+4}} = \dfrac{1}{\sqrt{34}}(-1,2,5,-2) = \left(\dfrac{-1}{\sqrt{34}}, \dfrac{2}{\sqrt{34}}, \dfrac{5}{\sqrt{34}}, \dfrac{-2}{\sqrt{34}}\right)$

7. $d(u,v) = \|u-v\| = \|(-1,-2,-1)\| = \sqrt{1+4+1} = \sqrt{6}$.