Applications of the Chain Rule

(§ 3.5, 3.6, 3.7, 5.3 revisited, 5.4 revisited)

I. Fundamental Theorem with Chain Rule (§ 5.4)

$$\frac{d}{dx}\int_a^x f(t)dt = f(x),$$

$$\frac{d}{dx}\int_x^a f(t)dt = -\frac{d}{dx}\int_a^x f(t)dt = -f(x),$$

$$\frac{d}{dx}\int_a^{h(x)} f(t)dt = \left(\frac{d}{du}\int_a^u f(t)dt\right)\left(\frac{du}{dx}\right) = f(u)u_t' = f(h(x))h'(x).$$

$$\frac{d}{dx}\int_{h(x)}^a f(t)dt = -\frac{d}{dx}\int_a^{h(x)} f(t)dt = -f(u)u_t' = -f(h(x))h'(x).$$

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = \frac{d}{dx}\int_a^{h(x)} f(t)dt + \frac{d}{dx}\int_{g(x)}^a f(t)dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Examples

1. $\frac{d}{dx}\int_0^x e^{2t}dt = e^{2x}$.

2. $\frac{d}{du}\int_u^{-1} e^{2x}dx = -e^{2u}$.

3. $\frac{d}{dx}\int_{\pi/2}^{\sin x}\cos t dt = \left(\frac{d}{du}\int_a^u \cos t dt\right)\left(\frac{d\sin x}{dx}\right) = \cos u \cos t = \cos(\sin x)\cos x$.

4. $\frac{d}{dx}\int_{\arctan x}^0 \sin t dt = -\frac{d}{dx}\int_0^{\arctan x}\sin t dt = -\sin(\arctan x)\frac{1}{1+x^2} = -\frac{x}{\sqrt{1+x^2}}\frac{1}{1+x^2} = -\frac{x}{(1+x^2)^{3/2}}$.

5. $\int_x^{x^2}\ln t dt = 2x\ln(x^2) - \ln x = (4x-1)\ln x$.

II. Implicit Differentiation (§ 3.5)

Implicit function: $F(x, y) = G(x, y)$.

Finding derivatives by implicit differentiation:

(i) Take the derivative on both sides **with respect to *x***. In this step, y is regarded as a function of y. Hence, the derivative of a function of y to x is found by the chain rule.

(ii) Solve the equation for y_x'.

Examples

(1) $x^2 + y^3 = 1$. $2x + 3y^2y' = 0$. $y' = -2x / (3y^2)$. When $x = 0$, $y' = 0$, when $y = 0$, $x = \pm 1$, $y' = 0$.

(2) $x^3 + y^3 - 3xy + 2x - 2y = 0$. $3x^2 + 3y^2y' - 3y - 3xy' + 2 - 2y' = 0$.
$y' = -(x^2 - y + 2) / (y^2 - x - 2)$. At (0, 0), $y' = 1$.

(3) $\sin(x + y) = 2x / y$. $[\cos(x + y)](1 + y') = 2(y - xy') / y^2$. $y^2\cos(x + y)(1 + y') = 2y - 2xy'$.

$y' = -[y^2\cos(x + y) - 2y] / [y^2\cos(x + y) + 2x]$.

At $(0, \pi)$, $y' = (\pi^2 + 2\pi) / (-\pi^2) = -(\pi + 2) / \pi$.

(4) $e^y - x = 0$. $y'e^y - 1 = 0$. $y' = 1 / e^y$. Since $y = \ln x$, $y' = 1 / e^{\ln x} = 1 / x$.

(5) Find the equation of the tangent line of the curve $\sin x + \cos y = \sin x \cos y$ at $(0, \pi / 2)$.

Find the derivative y_x': $\cos x - y' \sin y = \cos x \cos y - y' \sin x \sin y$. When $x = 0$, and $y = \pi / 2$, $1 - y' = 0$. $y'_x = 1$. The equation has the form $y = x + b$. When $x = 0$, $y = \pi / 2$. $b = \pi / 2$. Therefore, the equation of the tangent line is $y = x + \pi / 2$.

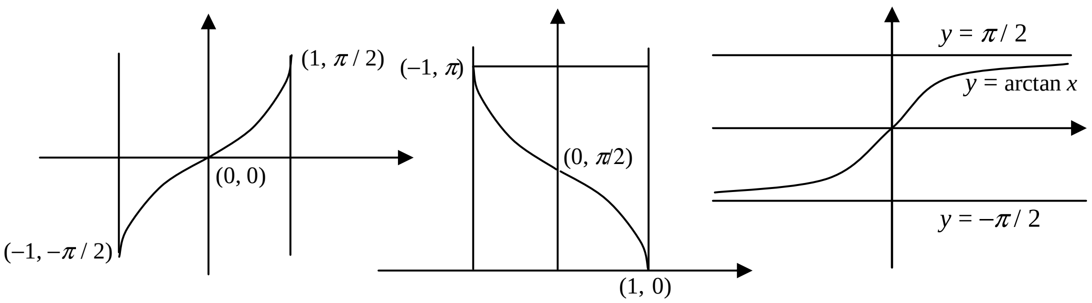
III. Inverse Trigonometric Functions and Their Derivatives (§ 3.6, 5.3)

1. *Inverse Trigonometric Functions*

- A function has an inverse if and only if it is one to one.
- Functions $y = \sin x$, $y = \cos x$, and $y = \tan x$ are NOT one to one.
- Restrict the domain of these functions to make "restricted" sin, cos, and tan function. They are one to one functions.
- Inverse trigonometric functions are inverses of "restricted" trigonometric functions.

Two notations: arcsin versus $\sin^{-1}$.

The graphs of arcsin, arccos, arctan:



Properties of inverse trigonometric functions from their graphs

Domain, range, monotonicity, concavity.

sin(arcsin x) = x but arcsin(sin x) is not necessarily x.
cos(arccos x) = x, but arccos(cos x) is not necessarily x.
tan(arctan x) = x but arctan(tan x) is not necessarily x.

cos(arcsin x) = $\sqrt{1-x^2}$.

tan(arcsin x) = $\dfrac{x}{\sqrt{1-x^2}}$.

sin(arccos x) = $\sqrt{1-x^2}$.

tan(arccos x) = $\dfrac{\sqrt{1-x^2}}{x}$.

sin(arctan x) = $\dfrac{x}{\sqrt{1+x^2}}$.

cos(arctan x) = $\dfrac{1}{\sqrt{1+x^2}}$.

Derivatives of Inverse Trigonometric Functions

(1) y = arcsin x

$\sin y = x$, $y' \cos y = 1$. $y' = 1 / \cos y$. When $-\pi / 2 \le y \le \pi / 2$, $\cos y \ge 0$.
$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$. $y' = \dfrac{1}{\sqrt{1-x^2}}$.

(2) y = arccos x

$\cos y = x$, $-y' \sin y = 1$. $y' = -1 / \sin y$. When $0 \le y \le \pi$, $\sin y \ge 0$, $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$.

$y' = -\dfrac{1}{\sqrt{1-x^2}}$.

(3) $y = \arctan x$

$\tan y = x$. $\dfrac{y'}{\cos^2 y} = 1$. $y' = \cos^2 y = 1 / (1 + \tan^2 y) = 1 / (1 + x^2)$.

Integration formulas

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \quad \text{(or, } -\arccos x + C\text{)}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Examples

(1) $y = x \arctan\sqrt{x}$. $y' = \arctan\sqrt{x} + x / [(1 + x)2x^{1/2}] = \arctan\sqrt{x} + \sqrt{x} / [2(1 + x)]$.

(2) $y = \tan(\arcsin x)$. $y' = \dfrac{1}{\cos^2(\arcsin x)}\dfrac{1}{\sqrt{1-x^2}} = \dfrac{1}{(1-x^2)^{3/2}}$.

(3) $y = \arcsin x + \arccos x$. $y' = 0$.

(4) $y = \dfrac{\arccos x}{\sqrt{1-x^2}}$, $y' = \dfrac{-\dfrac{1}{\sqrt{1-x^2}}\sqrt{1-x^2} - \arccos x \dfrac{-x}{\sqrt{1-x^2}}}{1-x^2} = -\dfrac{\sqrt{1-x^2} - x\arccos x}{(1-x^2)^{3/2}}$.

(5) $\displaystyle\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1}\frac{1-(1-x^2)}{\sqrt{1-x^2}}\,dx = [\arcsin x]_{x=-1}^{1} - \frac{\pi}{2} = \pi/2 - (-\pi/2) - \pi/2 = \pi/2$.

(6) $\displaystyle\int_{-1}^{1}\frac{x^3-x^2+x}{x^2+1}\,dx = \int_{-1}^{1}\left(x - 1 + \frac{1}{x^2+1}\right)dx = -2 + [\arctan x]_{x=-1}^{1} = -2 + \pi/4 - (-\pi/4) = \pi/2 - 2$.

IV. Derivatives of Logarithmic Functions (§ 3.7, 5.3)

$y = \ln x$, $x > 0$. $y' = 1 / x$. $y = \ln(-x)$, $y' = 1 / x$.

$y = \ln(-x)$, $x < 0$. $y' = -(1 / (-x)) = 1 / x$.

Putting together: $y = \ln|x|$, $x \neq 0$, $y' = 1/x$.

$y = \log_a x = \ln x / \ln a$, $y' = \dfrac{1}{(\ln a)x}$.

Integration Formula:

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

The number e as a limit:

$y = \ln x$, $y'(1) = 1 = \lim_{h\to 0}\dfrac{\ln(1+h)-\ln 1}{h} = \lim_{h\to 0}\ln(1+h)^h$. Hence, $\lim_{h\to 0}(1+h)^{1/h} = e^1 = e$. Or

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Examples

(i) $y = x(\ln x - 1)$. $y' = (\ln x - 1) + x(1/x) = \ln x$.

(ii) $y = \ln\left(x+\sqrt{x^2+1}\right)$. $y' = \dfrac{1}{x+\sqrt{x^2+1}}\left(1+\dfrac{1}{\sqrt{x^2+1}}\right) = \dfrac{1}{\sqrt{x^2+1}}$.

(iii) $\displaystyle\int \frac{(x-1)^2}{x}\,dx = \int (x-2+1/x)\,dx = \frac{x^2}{2} - 2x + \ln|x| + C$.

V. Logarithmic Differentiation

Logarithmic differentiation is use in two cases:

Case 1. The function is a complicated product and/or quotient:

Examples

(i) $y = x^{3/4}(x^2 + 1) / (3x + 2)^5$.

$\ln y = \ln x^{3/4} + \ln(x^2 + 1) - \ln(3x + 2)^5 = (3/4)\ln x + \ln(x^2 + 1) - 5\ln(3x + 2)$

$y'/y = (3/4)(1/x) + (2x)/(x^2 + 1) - 15/(3x + 2)$.

$y' = [x^{3/4}(x^2 + 1) / (3x + 2)^5][(3 / 4)(1 / x) + (2x) / (x^2 + 1) - 15 / (3x + 2)]$.

(ii) $y = \sin^2 x \tan^4 x / (x^2 + 1)^2$.

$\ln y = 2 \ln \sin x + 4 \ln \tan x - 2(x^2 + 1)$.

$y' / y = 2 \cos x / \sin x + 4 / (\cos^2 x \tan x) - 4x / (x^2 + 1)$.

$y' = [\sin^2 x \tan^4 x / (x^2 + 1)^2][2 \cos x / \sin x + 4 / (\cos x \sin x) - 4x / (x^2 + 1)]$.

(iii) $y = 3^{x^2-2x+2}5^{2x^2+3x}x$.

$\ln y = (x^2 - 2x + 2)\ln 3 + (2x^2 + 3x)\ln 5 + \ln x$

$= (\ln 3 + 2 \ln 5)x^2 + (-2\ln 3 + 3\ln 5)x + 2\ln 3 + \ln x = \ln\left(3\times 5^2\right)x^2 + \ln\left(\frac{5^3}{3^2}\right)x + 2\ln 3 + \ln x$.

$(\ln y)' = y' / y = (\ln 75^2)x + \ln\left(\frac{125}{9}\right) + \frac{1}{x}$.

$y' = 3^{x^2-2x+2}5^{2x^2+3x}x\left((\ln 75^2)x + \ln\left(\frac{125}{9}\right) + \frac{1}{x}\right)$.

Case 2. The function is of the form $f(x)^{g(x)}$.

(i) $y = x^x$. $\ln y = x \ln x$. $y' / y = \ln x + 1$. $y' = (\ln x + 1)x^x$.

(ii) $y = x^{\sqrt{x}}$. $\ln y = \sqrt{x}\ln x$. $y' / y = x^{-1/2}\ln x / 2 + x^{1/2} / x = (2 + \ln x) / (2\sqrt{x})$.

$y' = x^{\sqrt{x}}(2 + \ln x) / (2\sqrt{x})$.

(iii) $y = (\sin x)^x$. $\ln y = x \sin x$. $y' / y = \sin x + x \cos x$. $y' = (\sin x)^x(\sin x + x \cos x)$.

(vi) $y = x^{\ln x}$. $\ln y = (\ln x)^2$. $y' / y = 2 \ln x / x$. $y' = (2 \ln x)x^{\ln x} / x$.