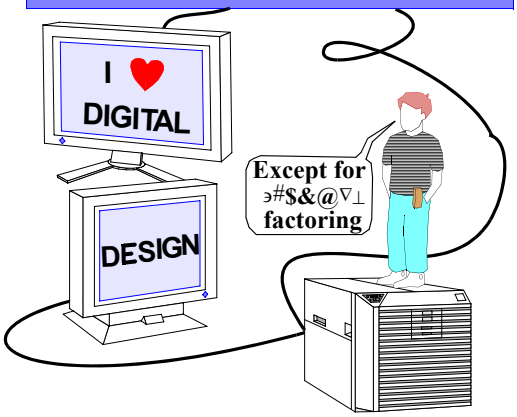
MULTIPLYING OUT & FACTORING



1st Distributive

$XA + XB = X(A + B)$

2nd Distributive

$(X + A)(X + B) = X + AB$

$(X + A)(X + B)(X + C) = X + ABC$

Swap

$(X + A)(\overline{X} + B) = XB + \overline{X}A$

The Most Common Stupid Errors

Using	*Not Using*
~~$X \cdot Y = \overline{X}\,\overline{Y}$~~	$YX + X = X$
~~$X + 1 = X$~~	$Y + X\overline{Y} = X + Y$

General DeMorgan

$\overline{F(a, b, \ldots z, +, \cdot, 1, 0)} \equiv F(\overline{a}, \overline{b}, \ldots \overline{z}, \cdot, +, 0, 1)$

Carleton University 2009

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Multiplying Out and Factoring

Sum-of-Products, Product-of-Sums

Multiplying Out

- Use D1
- Take dual; Simplify & D1 ; reverse dual; then use D1
- Use a Karnaugh Map

Factoring

- Use D2
- Take the dual ⇒ multiply out ⇒ take the dual back
- Find $\overline{F}$ using DeMorgan, then use a Karnaugh map

Standard Forms

Sum-of-Products Product-of-Sums

These are standard templates or *forms*

- **Every logical expression can be converted to either of these forms.**

Sum of Products

Σ **of** Π, **OR of ANDs, S of P**

- **Single variables ANDed together into terms.**
- **These terms are ORed together.**
- **Inversions are only over individual variables.**
- **No brackets.**

$$abc + \bar{c}de + ad + \bar{f} + db$$

Product of Sums.

Π **of** Σ, **AND of ORs, P of S**

- **Single variables ORed together into terms.**
- **These terms are ANDed together.**
- **Inversions are only over individual variables.**
- **Brackets only around variables in OR terms**

$$(a+b+c)(\bar{c}+d+e)(a+d)(\bar{f})(d+b)$$

Questions

Is ab + cd + b(d+e) Σ **of** Π **?**

Is (a +bc)(d+e) Π **of** Σ **?**

Terms

A term in a logical expression is a a number of single letters ANDed together. Examples:

abcd abcXq xy A

Depending on context, it could also represent a number of single letters ORed together. Examples:

a+b+c+Y+q A+B A

Sum-of-Products

These must be expressed algebraically as the OR of ANDs. One was not allowed to put in brackets, XORs, nor long inversion bars. Thus none of the examples on the left below are true sum of products.

NOT Σ **of** Π: ~~abc + ab(a+d) + de~~ ~~a(abc + bfg + bcd)~~ ~~$\overline{abc}$ + bcd +abf~~ ~~a⊕c + bg~~

d+abc
abc
a

These are Σ of Π

After finding an algebric Σ of Π form, it can be easily changed to NAND-NAND logicfor implementation.

Product of Sums

These must be expressed algebraically as the AND of ORs. One must use single-letters ORed together into terms which are ANDed together. There must be no XORs, nor long inversion bars. Thus none of the examples on the left below are true product of sum.

NOT Π **of** Σ: ~~(a+b+c)ab(a+d)~~ ~~(a+b+c)(d+e)+d~~ ~~$\overline{(a+b+c)}$(b+c+d)(a+b)~~ ~~(a⊕c+b)(g+h)~~

a(e+b+c)
a+b+c
a

These are Π of Σ

After finding a algebric Π of Σ form, it can be easily changed to NOR-NOR logic for implementation.

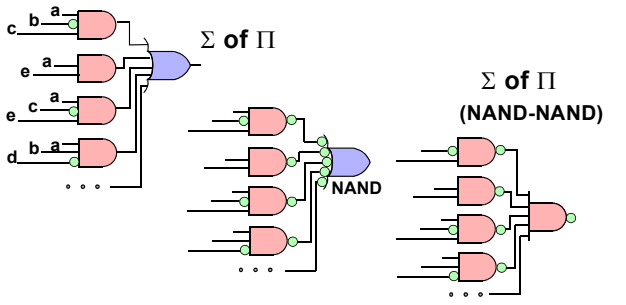
Two Standard Forms

Sum of Products (Σ of Π)

$\bar{a}bc + ae + a\bar{c}e + ab\bar{d} + \ldots$

OR of ANDs

Can be implemented as NAND-NAND logic



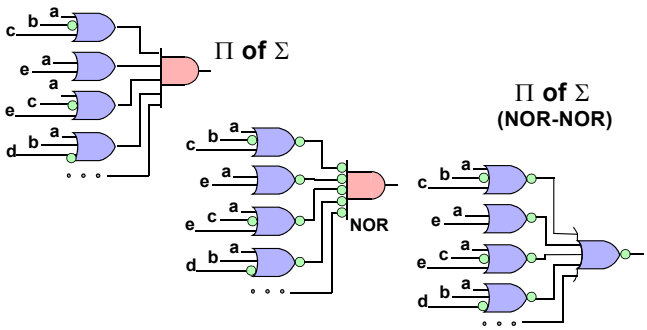
Product of Sum (Π of Σ)

Dual of Σ of Π

$(a+\bar{b}+c)(a+e)(a+\bar{c}+e)(a+b+\bar{d})(\ldots$

AND of ORs

Can be implemented as NOR-NOR logic



Standard Forms[1]

A Boolean expression can be represented in many ways, see below. A way to see if two .

There are many templates (*forms*) to define a Boolean Function.

A form that can define all Boolean functions is called a ***standard form.***

Some Standard Forms.

- The truth table is a very basic form
- A Σ of Π expression. → $\bar{a}b + a\bar{b}$
- A Π of Σ expression. → $(a + b)(\bar{a} + \bar{b})$ (factored form)
- Karnaugh maps

a	b	a⊕b
0	0	0
0	1	1
1	0	1
1	1	0

A form that will be introduced shortly.

- Binary decision diagrams, which will be used later to build arbitrary circuits with 2-input muxes.

Why we say Σ of Π for Sum of Product

Σ is used for a repetitive addition, as in $\sum_{i=0}^{n} x_i = x_0 + x_1 + x_2 + \ldots x_n$,

Π is used for a repetitive product, as in $\prod_{i=0}^{n} x_i = x_0 x_1 x_2 \ldots x_n$

[1] *A special type of standard form is the Canonical* form. It is one which is unique. If two apparently different Boolean expressions are reduced to canonical form, they are the same if-and-only-if their canonical forms are identical. Truth tables and unlooped Karnaugh maps are canonical standard forms. Σ of Π and Π of Σ are only standard forms
Canonical originally meant according to the rule or law, particularly the church law.

Factoring and Multiplying Out

Transforming Σ of Π ⟷ Π of Σ

Multiplying out

Transforms Π of Σ → Σ of Π

$$(a + c)(b + \overline{a} + d) \rightarrow ab + c\overline{a} + ad$$

Factoring

Transforms Σ of X → Π of Σ

$$ab + c\overline{a} + ad \rightarrow (a + c)(b + \overline{a} + d)$$

Why Factor?

1: Typically both forms have about the about the same size, but sometimes factoring can save significant logic.

Σ of Π	Π of Σ	Σ of Π	Π of Σ
$ab + ac + ad + ae$	$= a(b+c+d+e)$	5 gates, 8 letters	→ 2 gates, 5 letters
$acb + \overline{a}\cdot\overline{c}\cdot b + acd + \overline{a}\cdot\overline{c}\cdot d$	$= (a + c)(b + d)(\overline{a} + c)$	5 gates, 12 letters	→ 4 gates, 6 letters
$ac + ae + bc + be$	$= (a + b)(c + e)$	5 gates, 8 letters	→ 3 gates, 4 letters

2: Sometimes NOR-NOR logic may be desired (Fast fall time).

Σ of Π ⟷ Π of Σ

Why factor?

Usually the (Π of Σ) and (Σ of Π) forms are about the same complexity, but changing forms for part of the circuit can sometimes save significant logic.

Logic minimization programs will switch from one form to the other in different parts of a large circuit.

Π of Σ in NOR- NOR form, can give fast results with a falling signal 1=>0.[1]

4-1.• PROBLEM

Identify the following as: Σ of Π, Π of Σ, or neither.

1. $(\overline{W} + Y + \overline{Z})(X + Y + \overline{Z})(W + X)(W + Z)(X + \overline{Y} + \overline{Z})(W + X + \overline{Z})$
2. $(ab + c)(d + ce)(a + b + c + d)$
3. $dc + \overline{(abc)} + cb$
4. $abc + c(a+b) + dec + b$
5. $a + b + cd$

[1] NMOS transistors are faster than PMOS. A NOR gate use parallel NMOS transistors to lower its output signal. All other things being equal, this leads to slightly faster circuits.

Multiplying Out; Methods

Multiplying Out

Change Π **of** Σ $\rightarrow$ Σ **of** Π

(- - -)(- - -)(- - -) $\rightarrow$ ( ) + ( ) + ( )

1st Distributive Law (D1)

$x(A + B) = xA + xB$

Methods of Multiplying Out

I. Use the 1st distributive law

x(a + b) = xa + xb

- This always works, but
- it usually gives a very very long result.

II. Simplify (use D2, S2, Sw)
OR take dual (use D1, S, Sw) take dual before using D1

- Gives shorter answers
- Shorter steps to get answer
- Requires more thinking

III. Use DeMorgan's Law to get $\overline{F}$
Plot $\overline{F}$ on a K-map
Then plot F

- Easiest to do
- Gives the simplest Σ of Π answer.
- Gets messy for 5 or more variables.

Example I

$(A + B)(\overline{A} + C + D)$ *Use (D1)*

$= A\overline{A} + AC + AD + B\overline{A} + BC + BD$

$= AC + AD + B\overline{A} + BC + BD$ *Use* $A\overline{A} = 0$

Example II

$(A + B)(\overline{A} + C + D)$

$= (A + B)(\overline{A} + [C + D])$ *Use (Sw)* $(x+B)(\overline{x}+D) = xD + \overline{x}B$

$= A[C + D] + B\overline{A}$ *Use (D1)*

$= AC + AD + B\overline{A}$

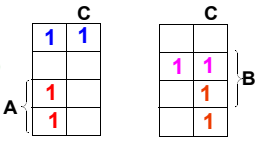
Example III

$F=(A + B)(\overline{A} + C)$

$\overline{F} = A\cdot\overline{B} + A\overline{C}$ *Use (DeM)*

Use Map

$F = B\overline{A} + AC$



$\overline{F}$ F

Three methods of multiplying out

Using D1

Using D1 many times is a straightforward way. D1 is very easy to use; unfortunately, when done automatically with no thinking, it can get very long. If one uses simplification (S) $X(X+a)=X$ and absorption (A) $X(\overline{X}+a)=Xa$ at every chance, the work is shorter, but the final expression still may be longer than necessary.

Hint: the simplify and absorb rules are easier to see in Σ of Π.

Take the dual so one uses $X+Xa=X$ and $X+\overline{X}a=X+a$ which are easier to see. Then take the dual back to use D1.

Use D2 (or equivalent) and Swap before using D1

The strange distributed rule, D2, is the dual of D1. The equivalent of using D2 is to: take the dual, use D1, then reverse the dual. Swap is the same difficulty in both the dual and in the original form, so use it any time. Following the hint above, take the dual and use (S) and (A) at the start. While in the dual, use D1. Then reverse the dual and continue using D1, remembering to watch for convenient places to use Swap, (S), and (A).

Using Karnaugh maps

This is the easiest method for four or five variables, it always gives the smallest answer, it easily handles don't cares, but gets very complex for over five input variables. **It is the method of choice for most small problems.**

Three methods of factoring

Using D2

This is the straightforward way, unfortunately it uses the "unfamiliar" D2 distributive law which makes the algebra harder for most people.

Using duality and D1

This is algebraically equivalent to the previous method. However using the more familiar (D1) makes it easier for most people.

Using Karnaugh maps

This is quite easy for four variables, but more complex for over five input variables. Easily handles "*d*"s.

Multiply Out Using 1st Distributive Law

Method I: Using D1

Step 1: Take Dual, Simplify
Step 2: Take Dual back
Step 3: Use D1
Step 2: Simplify, Use D1
Step 3: Simplify, Use D1
. . .
Final Step: Simplify, Simplify, (K-Map mat help)

1st Distributive Law
x(A + B) = xA + xB

Ways to Simplify
- X +aX = X (S)
- X +a$\overline{X}$ = X + a (A)
- Karnaugh map
- Consensus
  If too many inputs for map

Example

(A + X)(B + X)(C + X)
dual = AX + BX + CX = X(A+B+C)
f = ABC +X

= [A(B + X) + X(B + X)](C + X) *Use (D1)*
= [AB + AX + XB + X](C + X) *Use (D1)*
= [AB + X](C + X) *Use (S) x + ax = x*
= (AB + X)C + (AB + X)X *Use (D1)*
= (ABC + XC + ABX + X *Use (D1)*
= ABC + X *Use (S) x + cx = x*

Example of Multiplying Out

Simplification, absorption D2 (or equivalent) and Swap work well when their are repeated letters in the expression. However if all the letters are unique, can only use D1

$$(A + B)(C + D)(F + G + H)$$

All letters are different, no simplification possible

Use (D1)

$$= (AC + AD + BC + BD)(F + G + H)$$

rewrite

$$= \begin{cases} (AC + AD + BC + BD) \cdot F \\ + (AC + AD + BC + BD) \cdot G \\ + (AC + AD + BC + BD) \cdot H \end{cases}$$

Use (D1)

$$= \begin{cases} ACF + ADF + BCF + BDF \\ + ACG + ADG + BCG + BDG \\ + ACH + ADH + BCH + BDH \end{cases}$$

With all the letters different, there is no way to simplify.
The expressions get long rapidly.

Using (D1) always works, it is easy on the brain, but hard on the pencil.
Also the simplifications must be done by other means.

Multiply Out Using 1st Distributive Law

Method I: Using D1

Step 1: Simplify, Use D1
Step 2: Simplify, Use D1
Step 3: Simplify, Use D1
. . .
Final Step: Simplify

How to Simplify

1. $X + aX = X$ (S)
2. $X + a\overline{X} = X + a$ (A)
3. Karnaugh map
4. *Concensus if you have to*

1st Distributive Law (D1)

$x(A + B) = xA + xB$

Consensus

$Ab + bc + c\overline{A} = Ab + c\overline{A}$

Example:

$(B + C)(\overline{B} + F + C)(B + A)(\overline{B} + F + D)$

Use D1 twice

$= (\overline{B}B + BF + BC + C\overline{B} + CF + C)(B\overline{B} + BF + BD + A\overline{B} + AF + AD)$

$= (BF + C)(BF + BD + A\overline{B} + AF + AD)$ *Use D1*

$= BFBF + BFBD + BFA\overline{B} + BFAF + BFAD + CBF + CBD + CA\overline{B} + CAF + CAD)$

$= BF + CBD + CA\overline{B} + CAF + CAD)$

ANSWER, BUT COULD BE SIMPLER

Simplify using 5-input map but not done till next chapter

Can simplify using consensus twice but tricky!

$= BF + FAC + AC\overline{B} + BCD + ACD$

$BF + AC\overline{B}$ *consensus*

SKIP THIS TWADDLE

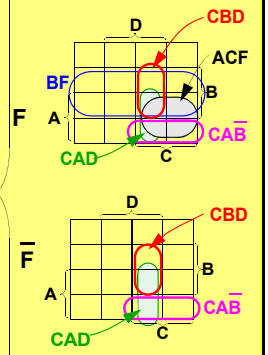
$= BF + C\overline{B}A + CAD + CDB$

$= BF + C(\overline{B}A + AD + DB)$

$C(BA + DB)$ *consensus*

$= BF + C\overline{B}A + CDB$

5-variable map



Result is $BF + CA\overline{B} + BCD$

Note on simplifying by consensus or a 5-input map (continued)

The 5-input map is not covered until the next section.
Using the 5-input map avoids the use of consensus in the above example. Consensus is tricky to see.

Example: Multiply Out using Method I. F is not a pure Π of Σ

Multiply Out

$F = [(A\cdot\overline{B} + \overline{A}\cdot\overline{D})] \cdot [\overline{A}\cdot(B + C) + A\cdot(\overline{C} + D)] \cdot (A + \overline{C} + \overline{D})$ *Use (D1) and* $x\overline{x}=0$

$= \{A\cdot\overline{B}\,[\overline{A}\cdot(B + C) + A(\overline{C} + D)] + \overline{A}\cdot\overline{D}\cdot[\overline{A}\cdot(B + C) + A(\overline{C} + D)]\} \cdot (A + \overline{C} + \overline{D})$ ($x\overline{x}=0$)

$= \{A\overline{B}\,[A(\overline{C} + D)] + \overline{A}\cdot\overline{D}\cdot[\overline{A}\cdot(B + C)]\} \cdot (A + \overline{C} + \overline{D})$ *Use (D1)*

$= \{A\overline{B}(\overline{C} + D) + \overline{A}\cdot\overline{D}\cdot(B + C)\} \cdot (A + \overline{C} + \overline{D})$ *Use (D1)*

$= \{A\overline{B}\cdot\overline{C} + A\overline{B}\cdot D + \overline{A}\cdot\overline{D}\cdot B + \overline{A}\cdot\overline{D}\cdot C\} \cdot (A + \overline{C} + \overline{D})$ *Use (D1)*

This psudo matrix helps arrange the messy multiplication

Use (D1) and $x\overline{x}=0$ ($x\overline{x} = 0$)

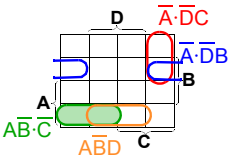
$= \begin{vmatrix} \{A\overline{B}\cdot\overline{C} + A\overline{B}\cdot D + \overline{A}\cdot\overline{D}B + \overline{A}\cdot\overline{D}C\}\cdot A \\ +\{A\overline{B}\cdot\overline{C} + A\overline{B}\cdot D + \overline{A}\cdot\overline{D}B + \overline{A}\cdot\overline{D}C\}\cdot\overline{C} \\ +\{A\overline{B}\cdot\overline{C} + A\overline{B}\cdot D + \overline{A}\cdot\overline{D}B + \overline{A}\cdot\overline{D}C\}\cdot\overline{D} \end{vmatrix}$ *Collect terms; use* $xy + x = x$

$= \begin{vmatrix} A\overline{B}\cdot\overline{C} + A\overline{B}D \\ + A\overline{B}\cdot\overline{C} + ABDC + \overline{A}\cdot DBC \\ + A\overline{B}\cdot\overline{C}\cdot\overline{D} \quad + \overline{A}\cdot\overline{D}B + \overline{A}\cdot\overline{D}C \end{vmatrix}$ ($xy + x = x$)

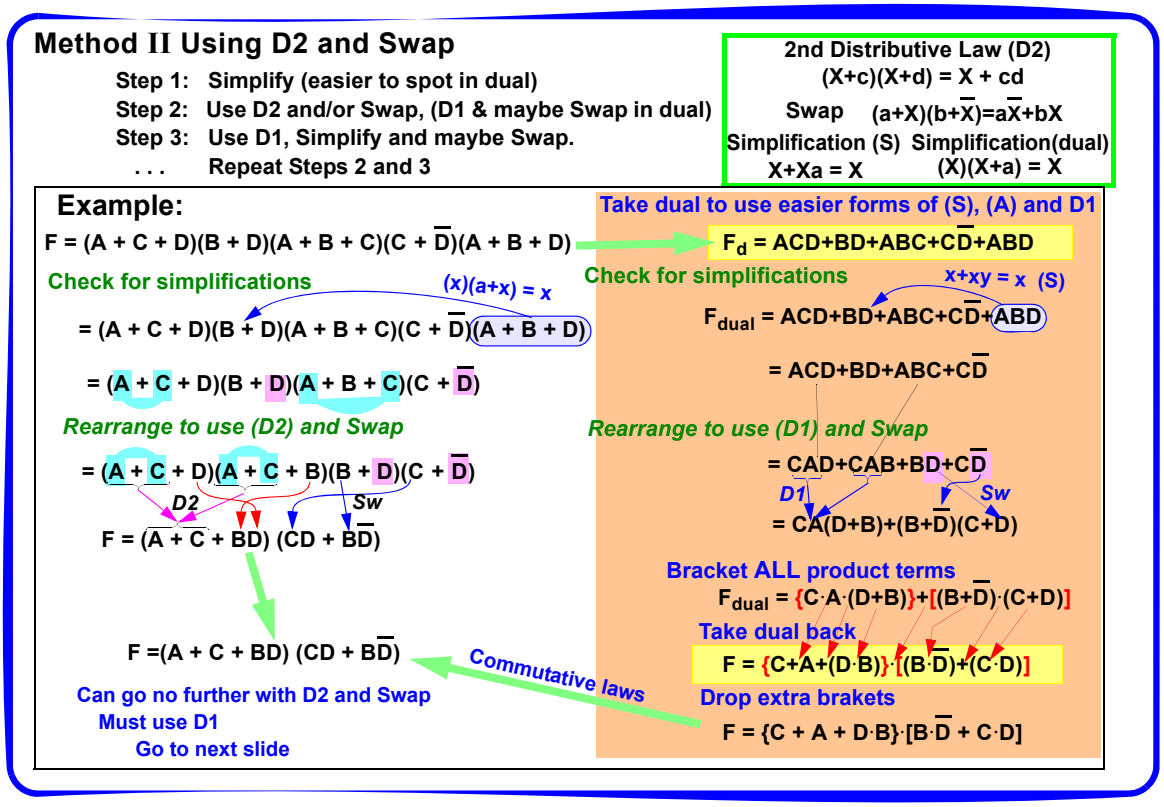
Plot map

$= A\overline{B}\cdot\overline{C} + A\overline{B}D + \overline{A}B\overline{D} + \overline{A}C\overline{D}$

Map shows there are no more simplifications



Multiply Out; Use D2 (or equ) and Swap, Before D1



Multiplying Out Using D2 and Swap before D1

Comments:

For the initial step

- "Simplify" here means using the dual rules X(X+a)=X and X(X̄+a)=X̄a
  However the rules X+Xa=X and X+X̄a=Xa are much easier to apply.
- Applying D2 here means (X+a)(X+b)=X+ab
  However D1 => Xa+Xb= X(a+b) is easier than using D2
- Swap is self dual and is equally obnoxious in both forms.

Instead of simplifying as above, take the dual as suggested in the brackets, and as is on the right side of the slide.

- Taking the dual of a Product-of-Sums is easy.
- After that one can use the easy simplify rules X+Xa=X and X+X̄a=Xa.
- One can also use the simple D1 instead of D2
- One can use swap here or in the next step, it makes little difference.[1]

The next slide multiplies out product terms using D1 => F_{dual} =(A + C + BD) (CD + BD̄)

- This is easier to do in the original form.
- Thus take the dual back before going on to slide 2.
  This dual is error prone. The trick is to be sure you put BRACKETS AROUND ALL THE **AND** TERMS!
  If you think you can write this on the same line as converting "+" ↔ "·", I have some land to sell you!

[1] Using swap while in the dual form introduces more brackets. More brackets increase the error tendancy when taking the reverse dual. Thus there is a slight advantage in using Swap later.

Multiply Out Using D2 and Swap, Before D1

Example from last slide (continued)

1st Distributive Law (D1)
$x(A + B) = xA + xB$

Original Problem

Multiply Out:

$$F = (A + C + D)(B + D)(A + B + C)(C + \overline{D})(A + B + D)$$

On last slide, using D2 (or equ) and Swap obtained

$$(A + C + BD)\,(CD + B\overline{D})$$

Use (D1)

$$= \begin{cases} (A + C + BD) \cdot CD \\ + (A + C + BD) \cdot B\overline{D} \end{cases} = \begin{cases} A{\cdot}CD + CD + B{\cdot}CD \\ + AB\overline{D} + CB\overline{D} + BD{\cdot}B\overline{D} \end{cases}$$

Use (D1) *x +xy = x*; $B\overline{D}$ term $= 0$

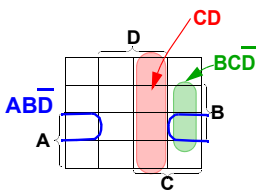
Collect terms

$$= CD + AB\overline{D} + BC\overline{D}$$

$$= CD + AB\overline{D} + BC$$

Check map for further simplifications

Are there any?



Example With Method II: Multiply out using D2 (or equ) and Sw before using D1

$$F = (A + \overline{D})(\overline{A} + \overline{C} + D)(\overline{A} + \overline{B})(A + B + C)(A + \overline{C} + \overline{D})$$

$F_d = (A{\cdot}\overline{D}) + (\overline{A}{\cdot}\overline{C}{\cdot}D) + (\overline{A}{\cdot}\overline{B}) + (A{\cdot}B{\cdot}C) + (A{\cdot}\overline{C}{\cdot}\overline{D})$ — *Take dual;*

$F_d = (A{\cdot}\overline{D}) + (\overline{A}{\cdot}\overline{C}{\cdot}D) + (\overline{A}{\cdot}\overline{B}) + (A{\cdot}B{\cdot}C) + (A{\cdot}\overline{C}{\cdot}\overline{D})$ — 0 — *use $A+AX = A$*

$F_d = (A{\cdot}\overline{D}) + (\overline{A}{\cdot}\overline{C}{\cdot}D) + (\overline{A}{\cdot}\overline{B}) + (A{\cdot}B{\cdot}C)$ — *Factor out A and then $\overline{A}$ using D1*

$F_d = A{\cdot}(\overline{D} + B{\cdot}C) + \overline{A}{\cdot}(\overline{B} + \overline{C}{\cdot}D)$ — *Use Swap*

$F_d = [A + (\overline{B} + \overline{C}{\cdot}D)]{\cdot}[\overline{A} + (\overline{D} + B{\cdot}C)]$

$F_d = [A + \overline{B} + \{\overline{C}{\cdot}D\}]{\cdot}[\overline{A} + \overline{D} + \{B{\cdot}C\}]$ — ***Bracket ANDed terms*** *(ready to take dual)*

$(\text{dual})^2 = F = [(A{\cdot}\overline{B}{\cdot}\{\overline{C}+D\}] + [\overline{A}{\cdot}(\overline{D}{\cdot}\{B+C\}]$ — ***Reverse dual***

$F = A{\cdot}\overline{B}{\cdot}\overline{C} + A{\cdot}\overline{B}{\cdot}D + \overline{A}{\cdot}\overline{D}{\cdot}B + \overline{A}{\cdot}\overline{D}{\cdot}C$ — *Use (D1)*

This reduces to the same function as the example on Slide 6

The Karnaugh map there shows there are no further simplifications for the Σ of Π form

$F = A\overline{B}{\cdot}\overline{C} + A\overline{B}D + \overline{A}B\overline{D} + \overline{A}C\overline{D}$

4-2.• PROBLEM

Multiply out. Remember to check for obvious simplifications before starting.

$(\overline{W} + Y + \overline{Z})(X + Y + \overline{Z})(W + X)(W + Z)(X + \overline{Y} + \overline{Z})(W + X + \overline{Z})$

Hint: Take the dual, simplify the dual and use D1 if possible, then take the reverse dual, before you multiply out.

4-3.• PROBLEM

Multiply out algebraically. The best answer has 8 letters. The hint from the last problem may be useful.

$(\overline{B} + \overline{C})(\overline{A} + \overline{B} + C)(\overline{A} + \overline{C} + \overline{D})(A + \overline{B} + C)(A + C + \overline{D})$

Multiply Out Using DeMorgan and K-Maps

Method III: Using K-Maps

Step 1: Find $\overline{F}$ using General DeMorgan
Step 2: Plot $\overline{F}$ on a K-Map
Step 3: Plot F using the 0 Squares
Step 4: Get expression for F from map

Example:

(1) Find the inverse using general DeMorgan

$F = (A + \overline{B} + C)(\overline{B} + \overline{C})(A + B + \overline{D})(C + \overline{D})$

$F_{dual} = (A\overline{B}C) + (\overline{B}\cdot\overline{C}) + (AB\overline{D}) + (C\overline{D})$

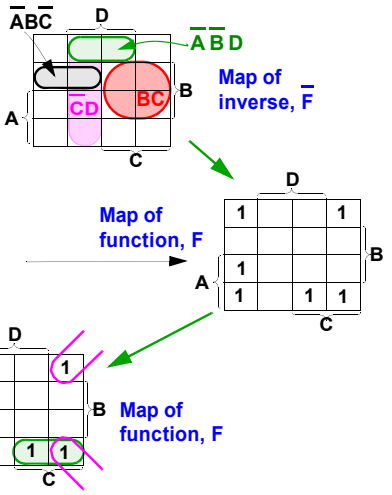
$\overline{F} = A\overline{B}C + BC + \overline{A}\cdot\overline{B}\cdot D + \overline{C}D$

(2) Plot $\overline{F}$ on a K-map

(3) Plot the F on a K-map; Put "1"s where $\overline{F}$ was "0"s.

(4) Circle F on K-map

$F = \overline{B}\cdot\overline{D} + A\overline{C}\cdot\overline{D} + A\,\overline{B}\,C$



The Easy Way to Multiply Out

$\overline{F}$ is easy to find using DeMorgan's law.
F is easy to find from a K-map of $\overline{F}$.

The above are the two essential facts used for multiplying out using a K-map.

When is the K-map the best method?

- For most problems done by hand is by far the easiest way to factor.
- Six or more variables will give a problem too big for a K-map.
- Using algebra may be easier if all the variables, or almost all, are different.See Comment on Slide 5.
- Algebra may be easier for converting small partial expression inside a long expression.

4-4.•PROBLEM:

Multiply out using DeMorgan's Law and a Karnaugh map, to get two terms of 2 letters and one of 3 letters.

$(A + C + D)(B + D)(A + B + C)(C + \overline{D})(A + B + D)$

4-5.•PROBLEM (Solution to a later factoring problem,and an earlier algebraic problem)

Multiply out to get an expression with eight letters. Use a Karnaugh map.

$(\overline{B} + \overline{C})(\overline{A} + \overline{B} + C)(\overline{A} + \overline{C} + \overline{D})(A + \overline{B} + C)(A + C + \overline{D})$

4-6.•PROBLEM (Remember Simplify, simplify, simplify!)

A multiply out problem using A,B,C,D,E and F as variables.

$(\overline{B} + \overline{C})(\overline{A} + \overline{B} + C)(\overline{E} + \overline{C} + \overline{D})(A + \overline{B} + C)(E + C + \overline{D})(\overline{C} + F)$

Multiply Out: Using DeMorgan and K-Maps

Example Using a Karnaugh Map

Steps:

(1) Given F = (Π of Σ expression)
F = (C + D)(A + $\overline{B}$ + $\overline{C}$)($\overline{A}$ + B + D)(A + B + C)

(2) Invert F using DeMorgan's law to get $\overline{F}$ as Σ of Π
$\overline{F}$ = $\overline{C}$·$\overline{D}$ + $\overline{A}$BC + A$\overline{B}$·$\overline{D}$ + $\overline{A}$·B·$\overline{C}$

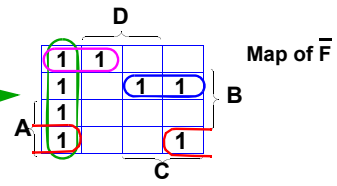
(3) Plot it on a map.

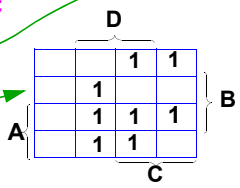
(4) Make a map for F, It has "1" where $\overline{F}$ had "0"

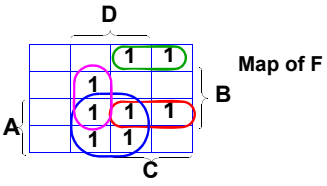
(5) Circle the F map

(6) Write out the equation for F

F = $\overline{A}$·$\overline{B}$C + AD + ABC + B$\overline{C}$D







Multiply Out With "*d*"s

If some input combinations are never used, these become don't care outputs.

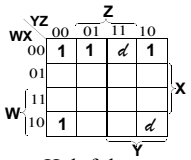
Do the normal steps up through finding $\overline{F}$
(1) Make the map for $\overline{F}$
(2) Make the map for F from that of $\overline{F}$.
Then identify the "d"s on the map of F.
Finally circle the map normally to find the minimum Σ of Π expression.

4-7.•PROBLEM:

Find the minimum Σ of Π expression for F using the don't cares to best advantage.

F = (W+X+Y)(W+X+Z)(X+Y+Z)

The input combinations $\overline{W}$·$\overline{X}$YZ, and W$\overline{X}$Y$\overline{Z}$ never happen, so these map squares are "*d*"s.



Helpful map

4-8.•PROBLEM:

Find the minimum Σ of Π expression for G using the don't cares to best advantage.

G = (C + D)(A + $\overline{B}$ + $\overline{C}$)($\overline{A}$ + B + D)(A + B + C)

The input combinations $\overline{A}$BCD, and $\overline{A}$·$\overline{B}$·$\overline{C}$D never happen.

Factoring

Factoring, the Dual of Multiplying Out

Change Σ of Π $\rightarrow$ Π of Σ

( ) + ( ) + ( ) $\rightarrow$ (- - -)(- - -)(- - -)

Methods of Factoring

I. Use the 2nd distributive law

$x + ab = (x + a)(x + b)$

- Always works, but
- very long and slow.

II. Take the dual
Use a multiply-out method
Take the dual back

- Changes unfamiliar D2 to familar D1
- In theory the same amount of work, but easier to grasp.

III. Plot F on a K-map
Plot $\overline{F}$ using the 0 squares
Find F using DeMorgan

- Easiest to do.
- Gives the simplest Π of Σ answer.
- Very messy for 5 variables or more.

Example I

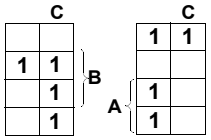
AC + AD + B
= (A+B)(C+B) + AD — *Use (D2)*
= [(A+B)(C+B) + A][(A+B)(C+B) + D] — *Use (D2)*
= [(A+B) + A][(C+B) + A][(A+B) + D][(C+B) + D] — *Use (D2)*
= [B + A][C+B+ A][A+B+ D][C+B + D] — *and a+a=a*
= [B + A][C+B + D] — *Use x(x+y) = x*

Example II

F = AC + AD + B
F = A(C + D)+B — *Use (D1)*
F = {A(C + D)}+B — *Take Dual*
F_{DUAL} = {A+C·D)}·B
F_{DUAL} = AB + CDB — *Use (D1)*
$(Dual)^2$ = F = (A+B)(C + D + B) — *Take Dual Back*

Example III

F = B$\overline{A}$ + AC — *Plot on Map*



F

$\overline{F}$

Get $\overline{F}$ from Map
$\overline{F}$ = $\overline{A}$·$\overline{B}$ + A$\overline{C}$ — *Use (DeM)*
F=(A + B)($\overline{A}$ + C)

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Four methods of factoring

I) Using D2

Using D2 many times is the brute force way. Unfortunately students find D2 hard to use, and the expansion may get very long. It helps to use simplification X (X+y) = X and absorption X ($\overline{X}$+y) = Xy at every chance, but these rules are also more difficult than their dual rules.

If all the letters are different, then all one can use is D2.

II) Using duality and D1

This in theory, is just as difficult as the previous method, but the more familiar rules makes it seem easier.

IIa) Using D1 a bit, then use duality and D1, as in II) above

D1 can do some factoring and often helps at the start. Example I above, using D1 before D2, can be done in two lines. AC+AD+B =A(C+D) +B (*using D1 first*) ={B+A}{B+(C+D)} (*using D2*)

III) Using Karnaugh maps

This is the easiest method for four or five inputs, it always gives the smallest answer, it easily handles don't cares, but it gets very messy for over five inputs. **It is the method of choice for most small problems.**

Three methods of multiplying out (compare with factoring)

Using D1

Using D1 many times is the straightforward way, D1 is very easy to use. Unfortunately the result can get very long. Using simplification (X + Xy =X) and absorption (X + $\overline{X}$y = X y) frequently will help.

Using D2 (or equ) and Swap before using D1

Use D2 or the equivalent (take the dual and use D1) and the Swap rule can do initial consolidation before using D1. In most cases one must use D1 for final cleanup.

Using Karnaugh maps

This is quite easy for four variables, but more complex for over five input variables. Easily handles "*d*"s

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Factoring Using D2

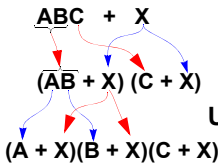
Method III: Using D2

2nd Distributive Law (D2)
$X + cd = (X+c)(X+d)$

Step 1: Simplify, Use D2
Repeat: Step 1 until done
. . .

Unless the problem is very simple the other methods will be easier.

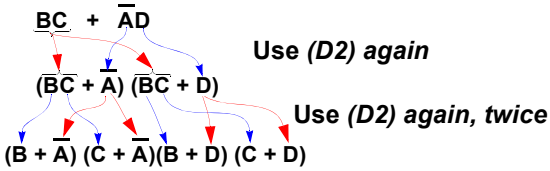
Example



Get extended *(D2)*

$ABC + X = (A + X)(B + X)(C + X)$

Example



Factoring Using *D2*

The expression proven on the slide is:

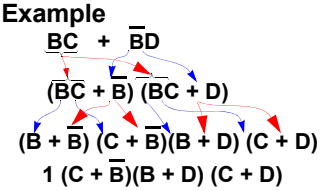
The extended (D2)

$ABC + X = (A + X)(B + X)(C + X)$

The dual is the extended (D1)

$(A + B + C)X = AX + BX + CX$

Example



(D2)

(D2) *(D2)*

$(B + \overline{B} = 1)$

If we had started by using the Swap rule, we would have a simpler answer in one step.

4-9.•PROBLEM

Factor AB + BC + CA

Factoring Using Duality

Method II: Factoring Using *Duality*

Step 1. Simplify and use D1 if possible

Step 2. Take the dual; get a factored, or semi factored form

Step 3. Multiply out the dual to get sum-of products.

Step 4. Take the dual back to get the factored form.

But for F_{dual}

The right one

Factoring Using Duality

The expression to factor is Σ of Π

$F = AB + BC + ACD$

(1) Use D1

$F = A(B + CD) + BC$

(2) Take its dual to get Π of Σ.

$F = [A\cdot(B + \{C\cdot D\})] + \{B\cdot C\}$ Bracket *ALL* AND terms

$F_{DUAL} = [A + (B\cdot\{C + D\})]\cdot\{B + C\}$

(3) Multiply out to get Σ of Π. See box

$F_{DUAL} = AB + BC + BD + AC$

(4) Take the dual back

$F = (B + A)(B + C)(B + D)(A + C)$

Get the desired Π of Σ.

$AB + BC + ACD = F = (B + A)(B + C)(B + D)(A + C)$

Multiply Out Details

$F_{DUAL} = [A + B\cdot\{C + D\}]\cdot\{B + C\}$

$= [A + BC + BD]\cdot\{B + C\}$ *(D1)*

$= [AB + BC + BD]$ *(D1)*

$+[AC + BC + BDC]$

bdX+bd=bd

$F_{DUAL} = AB + BC + BD + AC$

Multiplying out is based on (D1).

Easier for people, than factoring based on (D2).

Algebra of one is the dual of the algebra of the other

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Changing Factoring into Multiplying Out

Factoring is Converted to Multiplying Out, its Dual Problem

We take a factoring problem which is confusing, because factoring is based on (D2). This law is not a familiar "high-school type" algebraic law and is harder to work with.

In the dual space, the dual expression is already factored. The problem is transformed into multiplying out, which is based on the first distributive law (D1). (D1) is more familiar, and hence multiplying out is usually easier than factoring.

Multiplying out in the dual space does not give the answer. One take the dual to get the answer. This will then be the factored form of the original expression.

4-10.•PROBLEM

Show algebraically that

$$F = \overline{A}\cdot\overline{B}\cdot\overline{C}\cdot\overline{D} + \overline{A}B\overline{C}D + ABCD + A\overline{B}\overline{C}\overline{D}$$

takes only 8 letters or 12 gate inputs in factored form.

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Factoring Using Duality

Method II: Factoring Using Duality

Example:

$F = \overline{A}\cdot\overline{B}D + AB\cdot\overline{D} + \overline{A}BC + AC$

First use (D1)

$F = \overline{A}\cdot(\overline{B}D + B\overline{C}) + A(\overline{B}\cdot\overline{D} + C)$

$F = [\overline{A}\cdot(\{\overline{B}\cdot D\} + \{B\cdot\overline{C}\})] + [A\cdot(\{\overline{B}\cdot\overline{D}\} + C)]$

Take dual

$F_{dual} = [\overline{A} + (\{\overline{B} + D\}\{B + \overline{C}\})]\cdot[A + (\{\overline{B} + \overline{D}\}\cdot C)]$

Multiply out

equal

$= [\overline{A} + (\overline{B} + D)(B + \overline{C})]\,[A + (\overline{B} + \overline{D})C]$

Sw

$= \overline{A}\,(\overline{B} + \overline{D})C) + A(\overline{B} + D)(B + \overline{C})$

Sw

$= \overline{A}\,(\overline{B} + \overline{D})C) + A(\overline{B}\cdot\overline{C} + BD)$

$= \overline{A}\cdot\overline{B}C + \overline{A}\cdot\overline{D}C + A\overline{B}\cdot\overline{C} + ABD$

$F_{dual} = \overline{A}\cdot\overline{B}C + \overline{A}\cdot\overline{D}C + A\overline{B}\cdot\overline{C} + ABD$ *Check on map*

OK on map

Take dual back

$F = (\overline{A} + \overline{B} + C)(\overline{A} + \overline{D} + C)(A + \overline{B} + \overline{C})(A + B + D)$

Factored form -> $F = (\overline{A} + \overline{B} + C)(\overline{A} + \overline{D} + C)(A + \overline{B} + \overline{C})(A + B + D)$

First use (D1) twice

*Put brakets around **all** the AND terms*

Minus 25% if you don't differentiate between F and F_{dual}

Rearrange to use Swap

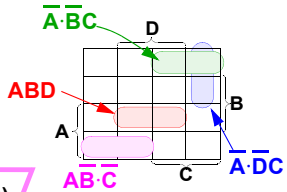
Use Swap

$(\overline{A}+stuff)(A+\overline{junk})=\overline{A}\cdot\overline{junk}+A\cdot stuff$

Use Swap

$(\overline{B}+D)(B+\overline{C})=\overline{B}\cdot\overline{C}+BD$

Use (D1)



map of F_{dual}

Method II: Factoring In the Dual Space

Example; Factor

$F = \overline{A}\cdot\overline{B}C + A\overline{C}D + ABC + B\overline{C}D$

$F = C(\overline{A}\cdot\overline{B} + AB) + \overline{C}(AD + BD)$ *Use (D1)*

$F = C(\overline{A}\cdot\overline{B} + AB) + \overline{C}\cdot D\cdot(A + B)$ *Use (D1)*

$F = [C + D\cdot(A + B)]\cdot[\overline{C} + (\overline{A}\cdot\overline{B} + A\cdot B)]$ *Use Swap*

In General Simplify, use D1 and maybe Swap before taking the dual

$F = [C + \{D\cdot(A + B)\}]\cdot[\overline{C} + (\{\overline{A}\cdot\overline{B}\} + \{A\cdot B\})]$ *Put brackets around all ANDs ready to take dual*

Take the dual

$F_d = [C\cdot\{D + (A\cdot B)\}] + [\overline{C}\cdot(\{\overline{A} + \overline{B}\}\cdot\{A + B\})]$

Remove extra brackets

$F_d = C\cdot\{D + (A\cdot B)\} + \overline{C}\cdot\{\overline{A} + \overline{B}\}\cdot\{A + B\}$

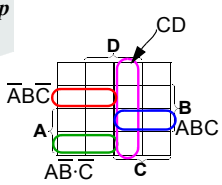
$F_d = C\cdot D + C\cdot A\cdot B + \overline{C}\cdot\{\overline{A} + \overline{B}\}\cdot\{A + B\}$ *Use (D1)*

$F_d = C\cdot D + C\cdot A\cdot B + \overline{C}\cdot[\overline{AB} + AB]$ *Use Swap*

$F_d = C\cdot D + C\cdot A\cdot B + \overline{C}\cdot\overline{AB} + \overline{C}\cdot AB$ *Use (D1)*

In General Simplify, use D1 and maybe Swap after taking the dual

$F_d = CD + CAB + \overline{C}B\overline{A} + \overline{C}\cdot BA + \overline{C}B\overline{A}$ *Map shows no more simplifications*



Take the dual back

$F = (C + D)\,(C + A + B)(\overline{C} + B + \overline{A})(\overline{C} + \overline{B} + A)$

4-11.• PROBLEM

Factor $E\overline{F}\cdot\overline{D} + ECD + \overline{E}\cdot\overline{C} + \overline{E}FD$

Factoring Karnaugh Maps and DeMorgan

Method III: Factoring Using a K-Map

Step 1. Plot function F on a K-map

Step 2. Plot $\overline{F}$ by interchanging 0 ↔ 1 on the map.

Step 3. Circle the map to get $\overline{F}$.

Step 4. Write out the expression for $\overline{F}$.

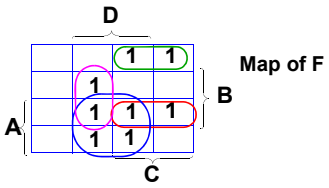
Step 5. Use DeMorgan to get back F in factored form.

Example:

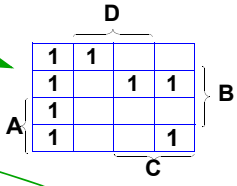
Given F = (Σ of Π expression)

$F = \overline{A}\cdot\overline{B}C + AD + ABC + B\overline{C}D$

(1) Plot it on a map.



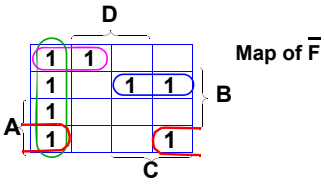
(2) Make a map for $\overline{F}$,
It has "1" where F had "0"



(3) Circle the $\overline{F}$ map

(4) Write out the equation for $\overline{F}$

$\overline{F} = \overline{C}\cdot\overline{D} + \overline{A}BC + A\overline{B}\cdot\overline{D} + \overline{A}\cdot B\cdot\overline{C}$



(5) Invert $\overline{F}$ using DeMorgan's law
to get F as Π of Σ

$F = (C + D)(A + \overline{B} + \overline{C})(\overline{A} + B + D)(A + B + C)$

Method III: Factoring Using Karnaugh Maps

- This method is probably the easiest, and least error prone, for up to four variables. Five variables is at least twice the work of four. Above 5 it gets very messy.
- It is very easy to incorporate don't cares with this method.

4-12.• PROBLEM

Factor $E\overline{F}\cdot\overline{D} + ECD + \overline{E}\cdot\overline{C} + \overline{E}FD$

Using a Karnaugh map and compare your answer with the previous problem if you did it.

4-13.•PROBLEM

Factor $ACD + BD + ABC + C\overline{D} + ABD$

Use a Karnaugh map and obtain the minimum Π of Σ expression.

4-14.•PROBLEM

Show, using a Karnaugh map, that

$F = \overline{A}\cdot\overline{B}\cdot\overline{C}\cdot\overline{D} + \overline{A}BC\overline{D} + ABCD + A\overline{B}C\overline{D}$

takes only 8 letters or 12 gate inputs in factored form. Compare with Problem 4-10.•

Factoring Using K-Maps and DeMorgan

Factoring Using Karnaugh Maps

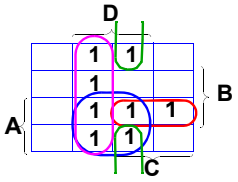
Steps:

Given F = (Σ of Π expression)

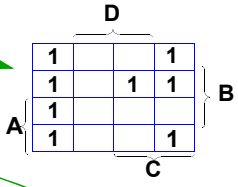
$F = \overline{A}\cdot CD + AD + ABC + \overline{C}D$

Not the minimum but it doesn't matter

(1) Plot it on a map.



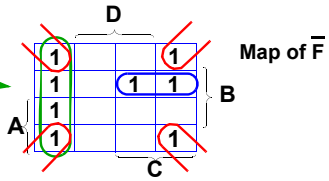
(2) Make a map for $\overline{F}$, It has "1" where F had "0"



(3) Circle the $\overline{F}$ map

(4) Write out the equation for $\overline{F}$

$\overline{F} = \overline{C}\cdot\overline{D} + \overline{A}BC + \overline{B}\cdot\overline{D}$



(5) Invert $\overline{F}$ using DeMorgan's law to get F as Π of Σ

$F = (C + D)(A + \overline{B} + \overline{C})(\overline{B} + \overline{D})$

Example: Factoring a 5-Variable Expression without using a map!

Method II: uses D1 initially, then duality and "swap" before the final cleanup D1.

Factor $F = A\overline{D} + \overline{A}\cdot\overline{C}D + \overline{A}\cdot\overline{B} + ABC + A\overline{C}\cdot\overline{E}$ *Use D1*

Use D1 $F = A(\overline{D} + BC + \overline{C}\cdot\overline{E}) + \overline{A}\cdot(\overline{C}D + \overline{B})$

Take dual

$F = [A\cdot(\overline{D} + \{B\cdot C\} + \{\overline{C}\cdot\overline{E}\}) + [\overline{A}\cdot(\{\overline{C}\cdot D\} + \overline{B})]$

$F_{dual} = [A + (\overline{D}\cdot\{B + C\}\cdot\{\overline{C} + \overline{E}\})]\cdot[\overline{A} + (\{\overline{C} + D\}\cdot\overline{B})]$

But brackets around ALL **AND** *terms*

$+ \leftrightarrow \cdot$

Use Swap

Multiply Out

$F_{dual} = [A\cdot\{\overline{C} + D\}\cdot\overline{B}] + [\overline{A}\cdot\overline{D}\cdot\{B + C\}\cdot\{\overline{C} + \overline{E}\}]$

Ready to use Swap

$F_{dual} = [A\cdot\{\overline{C} + D\}\cdot\overline{B}] + \overline{A}\cdot\overline{D}\cdot[\{C + B\}\cdot\{\overline{C} + \overline{E}\}]$

Use Swap

$[\{C\cdot\overline{E}\} + \{\overline{C}\cdot B\}]$

$F_{dual} = [A\cdot\{\overline{C} + D\}\cdot\overline{B}] + \overline{A}\cdot\overline{D}\cdot[\{C\cdot\overline{E}\} + \{\overline{C}\cdot B\}]$

Use (D1) twice

$F_{dual} = A\cdot\overline{C}\cdot\overline{B} + A\cdot D\cdot\overline{B} + \overline{A}\cdot\overline{D}\cdot C\cdot\overline{E} + \overline{A}\cdot\overline{D}\cdot\overline{C}\cdot B$

Take dual back get

$(dual)^2 = F = (A+\overline{B}+\overline{C})(A+\overline{B}+D)(\overline{A}+C+\overline{D}+\overline{E})(\overline{A}+B+\overline{C}+\overline{D})$

Warnings

Don't say
(a+b+c)(e+ab)(a+d)
is already Π of Σ

Always simplify. Look for $x+xZ = x, \quad x + \overline{x}Y = x + Y, \quad xY + \overline{x}Y = Y$ **before and after each step.**

Pick the best method:
A) for 5 variables or under use general Demorgan and a map.
B) for 6 variables or more, use algebra or find a computer program.

For multiplying out algebraically
Take the dual.
Simplify and use D1
Take the reverse dual.
Look for complemented letters, use swap, and simplify.
Use D1 for what is left.

For factoring algebraically
BEFORE you take the dual
Simplify and use D1
Take the dual use
Look possible Swap, simplify
Use D1 and simplify
Take the reverse dual

Taking the dual
The hard part is getting the brackets around the ALL the AND terms.

Try:
$A(\overline{D} + BC + \overline{C}\cdot\overline{E}) + \overline{A}(\overline{C}D + \overline{B})$

■ **Method III: Factoring Using Karnaugh**

Original the slide above said, "Don't say (a+b+c)ab(a+d) is already Π of Σ."

Actually it was Π **of** Σ, and the slide embarrased the lecturer.

4-15.• PROBLEM

Explain why the (a+b+c)ab(a+d) is Π of Σ. Hint: take the dual.

4-16.•PROBLEM ON DUALS

Take the dual of $F = A(\overline{D} + BC + \overline{C}\cdot\overline{E}) + \overline{A}(\overline{C}D + \overline{B})$

To check your answer look in the Comment on Slide 16
I'll bet you can't get it right the first time without looking.