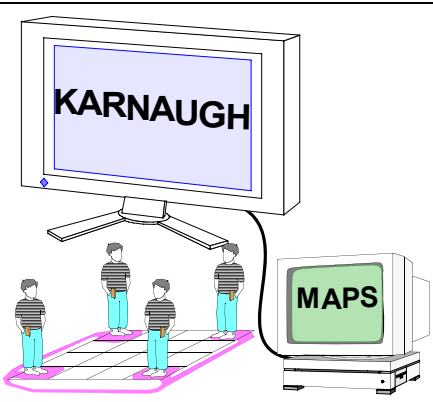


Digital Circuit Engineering



KARNAUGH

MAPS

Consensus, use Karnaugh

bc	0	1
a	00	01
	11	10
b	1	1

bc	0	1
a	00	01
	11	10
b	1	1

$ab + bc + c\bar{a} = ab + c\bar{a}$

~~$ab + bc + c\bar{a}$~~ \rightarrow $ab + c\bar{a}$

2nd Distributive

$(X + A)(X + B) = X + AB$

Simplification

$YX + X = X$

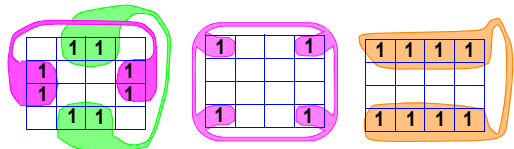
Absorption

$Y + X\bar{Y} = X + Y$

General DeMorgan

$\overline{F(a, b, \dots, z, +, \cdot, 1, 0)} \equiv F(\bar{a}, \bar{b}, \dots, \bar{z}, \cdot, +, 0, 1)$

Easily Forgotten Wrap-Arounds



Carleton University

2009



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Slide i

Chapter 3

Karnaugh Maps

Another form of the truth table

- Labelling Maps
- Circling Maps
- Minimizing Algebra by Maps
- Precautions when circling

Don't Cares

- Where don't cares come from
- Binary-Coded Decimal (BCD) digits
- Don't cares on Karnaugh Maps

Appendix

- XORs logic manipulation
- Using Karnaugh Maps with XORs

Karnaugh Maps; Equations From Truth Tables

A truth table with F not yet filled in.

abc	F
000	
001	
011	
010	
100	
101	
111	
110	

This order is important

Redraw table with halves side by side

half for a=0		half for a=1	
bc	F	bc	F
00		00	
01		01	
11		11	
10		10	

Compact the table

bc	F	F
00		
01		
11		
10		

Label inputs abc on the sides

bc	a=0	a=1
00		
01		
11		
10		

map of F

Check that moving one square only changes one input bit

1 bit changes

2 bits change

Layout for a Karnaugh map

Axis Labelling Conventions

Label abc on the sides

bc	a	0	1
00			
01			
11			
10			

Another way of labelling

a	a
$\bar{b}\bar{c}$	
$\bar{b}c$	
$b\bar{c}$	
$b\bar{c}$	

Abbreviated labelling

a

b { } c

Combined labelling

bc	a	0	1
00			
01			
11			
10			

b { } c

Karnaugh Maps; Equations From Truth Tables

Karnaugh Maps

Karnaugh Maps

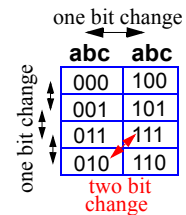
The map is like a truth table

Each square on the map represents a different input combination.
All possible input combinations are represented on the map.

The inputs are labelled around the edges of the map. Not inside the squares as shown on the right.

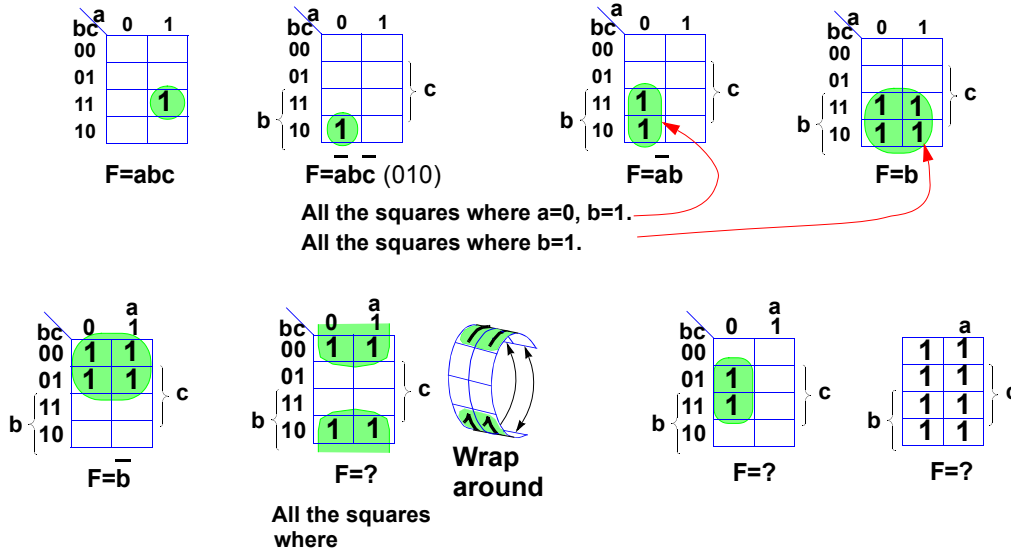
Arrangement of the squares

As one steps from one square to the next, either up, down, left or right, only one bit should change in a single step. If one goes to the nearest diagonal neighbour, two bits will change.



Karnaugh Maps; Representing AND Terms

Different Functions Using AND



Karnaugh Maps; Representing AND Terms ■

Representing AND Terms

Representing AND Terms

Any single square

(Top row, first two maps)

On these maps, any single square represents specific values for three variables, this is the same as a three term AND like \overline{abc}

Any two adjacent squares

We made the maps so that only one variable changes at a time if one moves vertically or horizontally. (This is not true for diagonal movements). Thus two adjacent squares always have one common variable.

In the top row, third map, the squares \overline{abc} and \overline{abc} are 1. We can say $\overline{abc} + \overline{abc} = \overline{ab}(c + \overline{c}) = \overline{ab}$

This shows that any two adjacent squares can be represented by a two term AND.

Any three adjacent squares

Meaning less!, One can only loop if the number of squares is a power of 2.

Any four adjacent squares

(Top row, fourth map)

There are two ways to look at this. One is that all the squares where $b=1$ have a "1" in them, hence one can describe them as b .

Alternately one can note that squares that are "1" are $\overline{abc} + \overline{abc} + abc + abc = b(\overline{a}\overline{c} + \overline{a}c + a\overline{c} + ac) = b$.

(bottom row, first two maps)

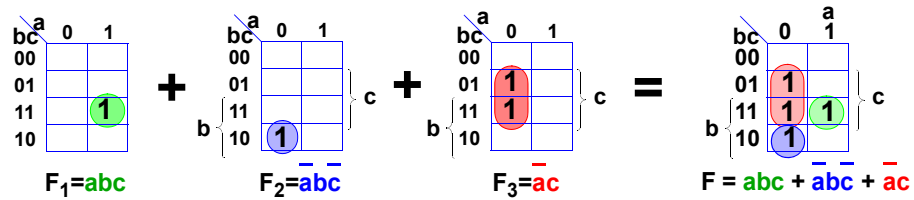
These represent \overline{b} and \overline{c} respectively.

Any eight adjacent squares

If all the squares on a map are "1", the function is $F=1$.

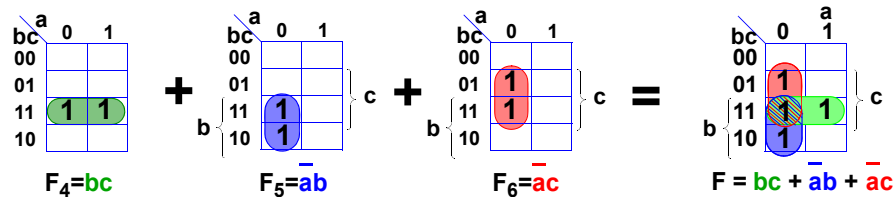
Karnaugh Maps; Joining AND Terms With ORs

Joining $F = F_1 + F_2 + F_3$ on the map



OR together the AND terms, and place them on one map.

The terms can overlap.



Using the larger terms (F_4 , F_5 , and F_6) gives a smaller expression for F .
Bigger loops give smaller gates.

Karnaugh Maps; Joining AND Terms With ORs

Circling a Map in Different Ways

■ Circling a Map in Different Ways

Combining Maps

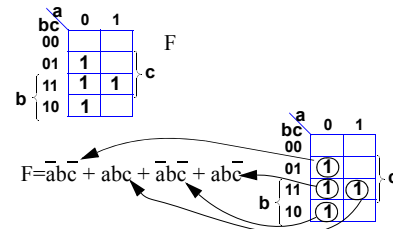
The OR of two maps, is a new map in which the squares are made 1 if there is a 1 in either (or both) of the initial maps

Same Map, Different Expressions

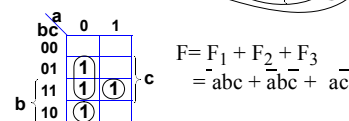
Take the truth table for a Boolean function F , written as a map.

One can loop it in several ways.

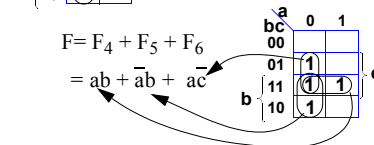
First one can loop individual "1"s. This gives a long expression for F .



Another way is to break it up as F_1 , F_2 and F_3 as shown on the top line on the slide.



A third way is to break it into F_4 , F_5 , and F_6 , as shown on the bottom line in the slide. This gives a smaller equation.



Maps for 1, 2, 3, 4, 5 and 6 Inputs; Legal Loops

Maps for different numbers of input variables

One

a	
0	
1	

Two

a	b	0	1
0			
1			

Three

bc	a	0	1
00			
01			
11			
10			

Four

cd	d	00	01	11	10
ab					
00					
01					
11					
10					

Five

Six

Allowed Loops for Simplifying Logic
 Must Loop adjacent squares
 Must loop 1,2,4,8,16 ... squares (ones)

Diagonal squares not adjacent

Looping with wrap around

Maps for 1, 2, 3, 4, 5 and 6 Inputs; Legal Loops

Karnaugh Map Properties

■ Karnaugh Map Properties

Maps may have any number of variables, but-

- Most have 2, 3, 4, or 5 variable. One variable is simple, (2 squares).
- five variables has two 4x4 maps, one for when e=1, and one for when e=0.
- Six variables have 64 squares and were used in pre-computer days.
- Seven variables is past the limit of sanity. You will have 8 blocks of 16 squares each.

Rules for looping "1"s

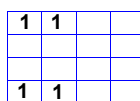
- "1"s on the maps can be looped to simplify logic.
- Only adjacent squares can be looped.
Diagonally adjacent squares are not considered adjacent.
- loops must surround 1, 2, 4, 8, 16 ... squares. Not 3,5,6,7,9,10, ...
- The maps wrap around.
A square on an edge is adjacent to the square on the opposite edge in the same column (row).

Larger loops give simpler logic, but-

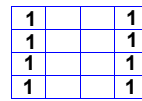
- One must obey the above rules.
- There are exceptions, particularly with multi-output maps.

3-1. PROBLEM

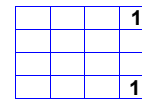
loop the "1"s on the three maps shown.



a)



b)

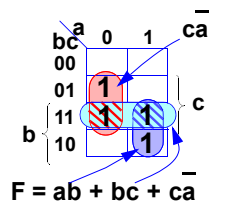


c)

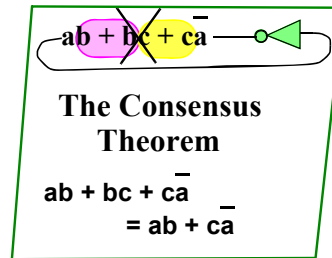
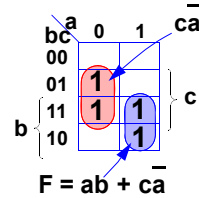
Karnaugh Maps; Simplifying Equations

Looping to Minimize the Logic Expression

Simplify $ab + bc + c\bar{a}$

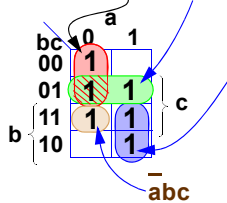


The bc term is redundant

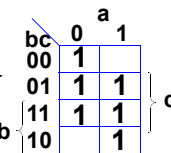


OR together the terms, and place them on one map.

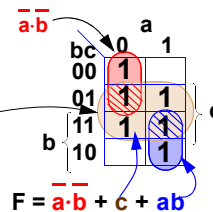
Simplify $F = \bar{a}\bar{b} + \bar{b}c + \bar{a}bc + ab$
 $= \bar{a}\bar{b} + \bar{b}c + \bar{a}bc + ab$



These are the "1s" to loop



Use a bigger loop in the middle



Using the larger terms (loops) gives a smaller expression for F.



Simplification of Boolean Function

A Boolean function can be defined in many ways

- A truth table
- A circuit diagram.
- A Karnaugh map without loops
- A looped Karnaugh map.
- A Σ of Π expression.¹
- A Π of Σ expression.
- A binary decision diagram (BDD).
- etc.

Any function has only one truth table and only one unlooped map.

There are usually several ways of circling the map or writing the algebraic expression for the same function.

One tries to find the best definition for some objective.

Possible Objectives

Smaller circuitry.

Lower powered circuitry.

Faster circuitry.

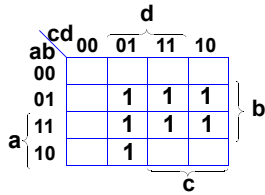
Making the circuit smaller is usually a good start toward making the circuit faster and lower powered.

¹. Σ of Π , Π of Σ , and BDD are defined in a later chapter, or try the glossary in Moodle.

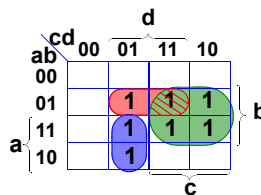
Karnaugh Maps; Best Loops to Reduce Logic

Simplification With 4-Input Maps

Simplify the function defined by this map

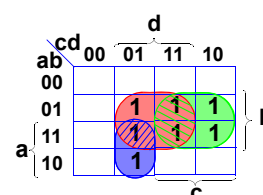


John's Solution



$$F = \bar{a}c\bar{d} + \bar{a}bd + bc$$

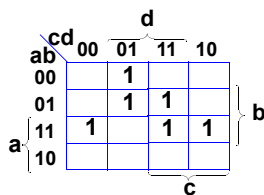
Tom's Solution



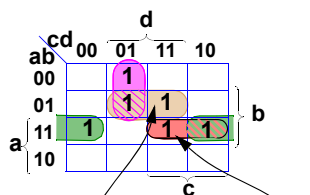
$$F = \bar{a}c\bar{d} + bd + bc$$

Usually bigger is better for loops

Simplify the function defined by this map

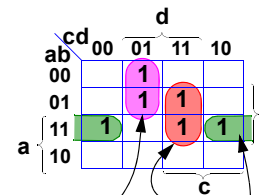


John's Solution



$$F = \bar{a}bd + \bar{a}cd + \bar{a}bd + abc$$

Tom's Solution



$$F = \bar{a}cd + bcd + \bar{a}bd$$



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Slide 7

Karnaugh Maps; Best Loops to Reduce Logic ■

Simplification

Simplification

Some Things to Do

Use the largest loop possible

John used $\bar{a}bd$ when he should have used bd .

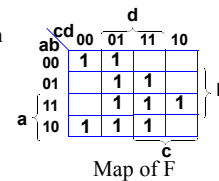
Check that overlap reduces, rather than increases logic.

John has two overlapping loops. Tom avoided both and saved a loop.

3-2. PROBLEM

Find the simplest expression for the logic function F^1 defined by looping the Karnaugh map on the right.

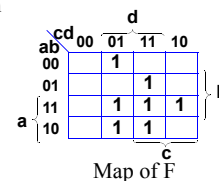
Get F with 9 letters. If it takes more, do Prob 3-1.a.



3-3. PROBLEM

Find the simplest expression for the logic function F defined by looping the Karnaugh map on the right.

Get F with 11 letters.

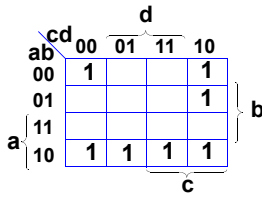


¹This logic function, written by ANDing together letters into terms, And ORing the terms, is called a *sum-of-products*

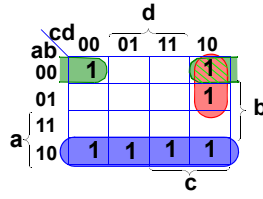
Karnaugh Maps; Best Looping to Reduce Logic

4-Input Maps, Looping Four Corners

Simplify the function defined by this map

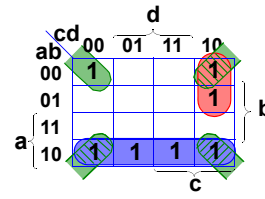


John's Solution



$$F = a \cdot \bar{b} + \bar{a} \cdot \bar{c} \bar{d} + \bar{a} \cdot \bar{b} \cdot \bar{d}$$

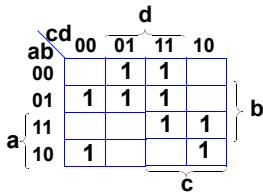
Tom's Solution



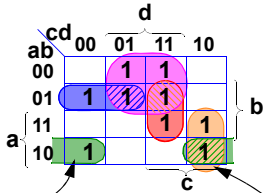
$$F = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c} \bar{d} + \bar{b} \cdot \bar{d}$$

Don't forget 4 corners

Simplify the function defined by this map

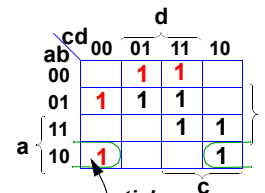


John's Solution



$$F = \bar{a} \cdot \bar{b} \cdot \bar{d} + \bar{a} \cdot \bar{b} \cdot \bar{c} + \bar{b} \cdot \bar{c} \bar{d} + \bar{a} \bar{d} + \bar{a} \bar{b} c$$

Your Solution



$$F = \bar{a} \cdot \bar{b} \cdot \bar{d} +$$



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Slide 8

Karnaugh Maps; Best Looping to Reduce Logic

Simplification

Simplification Theory

Essential Terms¹

A term (letters ANDed together that can be expressed by one loop) is essential if it contains at least one "1" that cannot be looped by any other loop, of the same or larger size.

"Your solution,"

The term $\bar{a} \cdot \bar{b} \cdot \bar{d}$, is *essential* in that no loop (except smaller ones) will cover the square $abcd=1000$.

$\bar{a} \cdot \bar{b} \cdot \bar{c}$, is *essential* in that no other loop (except smaller ones) will cover $abcd=0100$.

$\bar{a} \bar{d}$, is *essential* in that no other loop (except smaller ones) will cover $abcd=0001$ and 0011 .

Any logic function derived by looping a map must contain these *essential* terms. There is no choice, so loop them first.

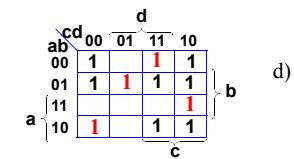
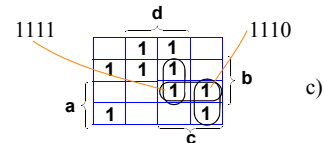
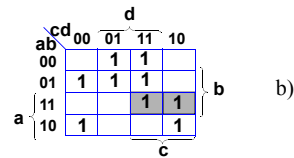
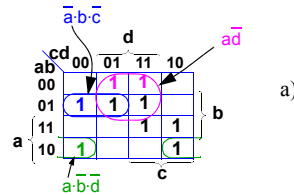
Squares Not Covered by Essential Terms

All the "1" squares except 1111 and 1110 are covered by the essential terms. One has a choice for terms to cover these squares.

There are three terms that cover one or both of these squares. Choose the ones to give the simplest expressions.

3-4. PROBLEM

Find the essential terms for each of the 4 colored "1"s on map d). Then find the simplest logic expression found by looping map d).



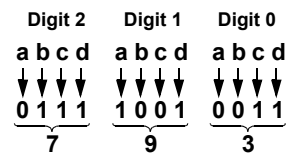
¹The more rigorous definition is *essential prime product term*, or *essential prime implicant*. Prime means the loop is maximum size.

Example of How One Gets Don't Care Outputs

Binary Coded Decimals (BCD)

decimal digit	Binary representation abcd	decimal digit	Binary representation abcd	not used	Binary representation abcd
0	0000	5	0101	x	1010
1	0001	6	0110	x	1011
2	0010	7	0111	x	1100
3	0011	8	1000	x	1101
4	0100	9	1001	x	1110
				x	1111

Representing a 3-digit number in decimal with 12 bits.



Map showing the values of abcd for each square

	cd	00	01	11	10
ab	00	0000	0001	0011	0010
	01	0100	0101	0111	0110
	11	1100	1101	1111	1110
	10	1000	1001	1011	1010

Map showing the decimal equivalent of the input bits

		d			
	cd	00	01	11	10
ab	00	0	1	3	2
	01	4	5	7	6
	11	x	x	x	x
a	10	8	9	x	x
		c			

"x" squares can never happen for BCD digits.



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Slide 9

Example of How One Gets Don't Care Outputs

Binary-Coded Decimals

Binary-Coded Decimals

These are used mainly for sending numbers to displays which people have to read.

Many years ago they were used to do commercial arithmetic. The story was that converting decimal fractions to binary caused small errors which could accumulate and throw off your bank account.

For example \$0.70 (decimal) = \$0.1110,0110,0110,0110,0110,0110,0110,0110, ... (binary)

Binary-coded decimal digits use 4 bits.

The table above shows that six of the sixteen 4-bit combinations are unused.

If one has a circuit which has binary-coded decimal inputs there will be six input combinations which never happen. These are marked with "x".

If they never happen, then one does not have to worry about the circuit's output for these input combinations.

These combinations are called *don't care* inputs and can be used to simplify the circuit.

3-5. PROBLEM

Write the year in binary coded decimal. In 2009, you should have 16 bits.

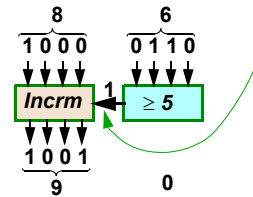
Don't Care Outputs; Where They Come From

Rounding BCD numbers

Round 2-digit BCD numbers to 1-digit.

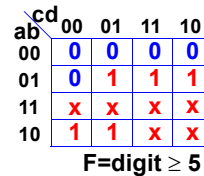
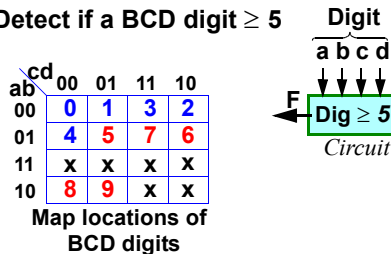
- 83 round to 80
- 86 round to 90
- 85 round to 90 (arbitrary choice)

If least sig digit ≥ 5
 - Send increment sig to next dig



Design Circuit

Detect if a BCD digit ≥ 5



red "1"s show digits ≥ 5

Don't Care Outputs; Where They Come From ■

Rounding BCD numbers.

Rounding BCD numbers.

There are three parts to this problem:

- (1) Design a circuit to check if a BCD digit ≥ 5 .
- (2) Modify an adder circuit to increment a BCD digit.
- (3) Put it all together.

We will only do part (1) in the slide above. Part 2 is a problem below.

Rounding Using the Don't Cares

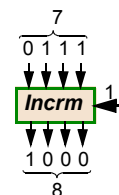
3-6. PROBLEM (Fairly difficult)

- a) Design the part (2) of the rounding circuit. A circuit that increment a BCD digit
 - One only needs half-adders, not full adders.
 - there is a carry in from the rounding signal.

If the digit coming into the incrementer was 9, incrementing it will overflow. For simplicity let

$$1001 + 1 = 1010$$

- b) A real BCD incrementer would increment 9 to give 0 with an overflow output signals. Add a circuit to check for a 1001 and an increment input, and give a 0000 plus an overflow.



Karnaugh Maps: Using Don't Cares

Don't Cares In Karnaugh Maps

Detect if a BCD digit is 5 or more.

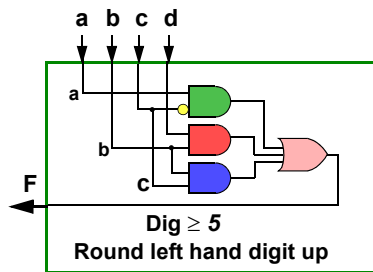
Use of "Don't Cares".

The BCD digits > 9 never happen
We don't care about output for them.
Make these outputs "d" on the map.

"d" may be looped or not as desired.

Loop to minimize logic
Here we looped 4 out of 6 "d"s, to get

$$F = \bar{a}c + db + bc$$

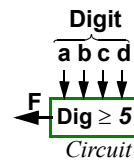


	cd	00	01	11	10
ab	00	0	1	3	2
	01	4	5	7	6
	11	x	x	x	x
	10	8	9	x	x

Map locations of BCD digits

	cd	00	01	11	10
ab	00	0	0	0	0
	01	0	1	1	1
	11	x	x	x	x
	10	1	1	x	x

F = digit ≥ 5



	cd	00	01	11	10
ab	00	0	0	0	0
	01	0	1	1	1
	11	d	d	d	d
	10	1	1	d	d

What's with this?
See notes below

	cd	00	01	11	10
ab	00	0	0	0	0
	01	0	1	1	1
	11	1	1	1	1
	10	1	1	0	0

Inputs to cause the six "d" outputs never happens, but if they did, with the this looping:

- the 4 looped outputs would then be "1", and
- the 2 unlooped ones would be "0".

Karnaugh Maps: Using Don't Cares ■

Don't Cares In Karnaugh Maps

Don't Cares In Karnaugh Maps

If one has input combinations that never happen, one does not care what outputs they generate because those outputs can never happen.

We put don't cares on the map squares for input combinations that never happen.

One can loop these don't cares or not as convenient.

There is a common error on the slide.

The loop $\bar{a}c$ could be extended to include all of \bar{a} .

This would give the final equation as

$$F = \bar{a} + bd + bc$$

3-7. PROBLEM (no ds)

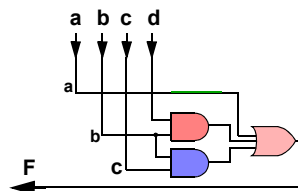
Take the hex digits 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F.

Plot their location on a Karnaugh map in the same way the BCD digits were plotted.

Then design a logic circuit which will use four bits w,x,y,z (defining a hex digit as input, and give a high output if the digit is divisible by 3, i.e. it is 3, 6, 9, C or F.

3-8. SUPPLEMENTAL PROBLEM (with ds)

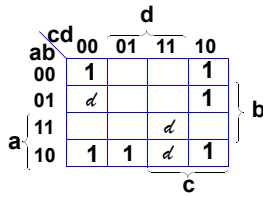
Design a circuit which will use four bits a,b,c,d defining a BCD digit as input, and gives a high output if the digit is divisible by 3. Utilize the inputs that cannot happen, to give don't care outputs, and hence simplify the logic.



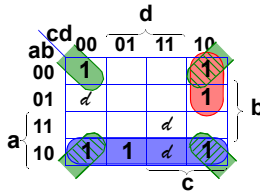
Karnaugh Maps; Examples With Don't Cares

Simplification With Don't Cares

Simplify the function defined by this map



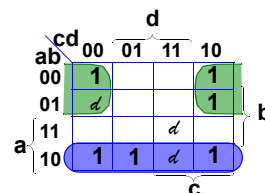
John's Solution



$$F = \bar{a}\bar{b} + \bar{a}c\bar{d} + \bar{b}\bar{d}$$

Don't have to loop all the "d".

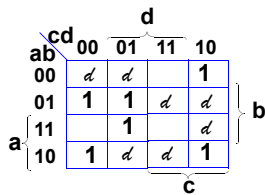
Tom's Solution



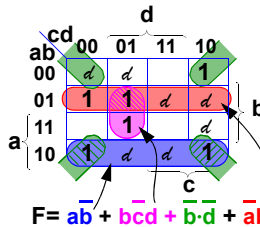
$$F = a\bar{b} + \bar{a}\bar{d}$$

4 corners not so good

Simplify the function defined by this map

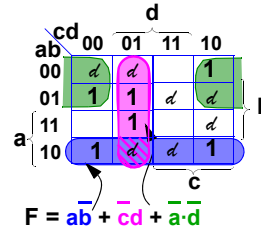


John's Solution



$$F = \bar{a}b + \bar{b}cd + \bar{b}\bar{d} + \bar{a}\bar{b}$$

Tom's Solution



$$F = \bar{a}b + \bar{c}d + \bar{a}\bar{d}$$



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Karnaugh Maps; Examples With Don't Cares ■

Don't Cares In Karnaugh Maps

3-9. PROBLEM: K-MAP WITH DON'T CARES

One way to design a comparator for two binary numbers is shown. It is made of blocks, each of which compares two bits. The example compares two, 3-bit, binary numbers $X=x_2x_1x_0$ and $Y=y_2y_1y_0$. It uses three blocks.

Each block compares inputs x_i with y_i with the inputs A_i, B_i from the comparison done for higher order bits. The result give outputs A_{i-1} and B_{i-1} .

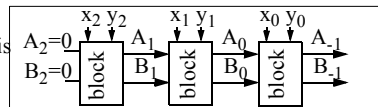
Thus if $A_i=1$, coming into a block, then the left hand bits have shown $x_2x_1x_0 > y_2y_1y_0$ without even bothering to check x_i and y_i . Similarly any if $B_i=1$ then $y_2y_1y_0 > x_2x_1x_0$.

A typical block is shown. We drop the cumbersome subscripts and write A^-, B^- to tell the A, B inputs from the outputs.

The "-" in the 5th line of the truth table inputs means: If $A, B = 1, 0$ then, no matter what x and y are, the output is $A^-, B^- = 1, 0$.

Do not confuse these with the don't cares in the outputs, "d", which are the result of input combinations that never happen.

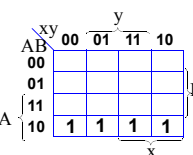
Complete the Karnaugh map for A^- including the don't cares. Then deduce the expression for A^- which should have 4 letters.



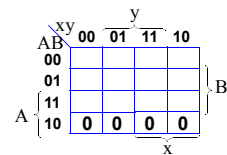
If $A=10$, then the left hand bits have already shown $x_2x_1x_0 > y_2y_1y_0$ independent of x and y . Thus the block should send out $A^-=1$. Since one cannot have both numbers larger, $B^-=0$.

x	y	A^-	B^-
A	B		
1	0	1	0
0	1	0	1
1	1	1	1
0	0	0	0

A	B	x	y	A^-	B^-
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	1	1
1	0	-	-	1	0
0	1	-	-	0	1
1	1	-	-	d	d



Map of A^-



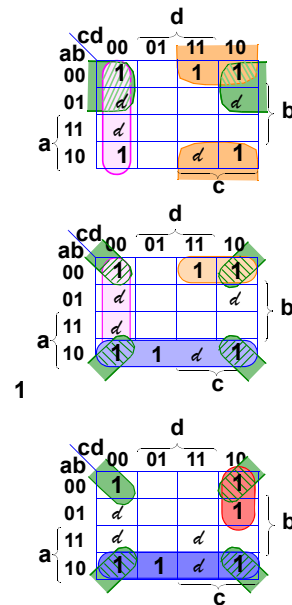
Map of B^-

Maps With Don't Cares; Common Mistakes

Don't Cares

Common mistakes

1. Looping don't cares when there is no need. Remember you loop them only if convenient. (*top*)
2. Over looping. (*mid*)
3. Forgetting wrap-around. (*mid, bot*)
4. Not enclosing don't cares which would make the loops larger. (*mid, bot*)



Common Mistakes with Don't Cares

Top: The four corners would be better. There is no need to include the upper two "d"s.

Middle: The four corners are redundant. Also the top orange $\bar{a}\bar{b}c$ oval could be doubled with wrap around.

Bottom: The red oval $\bar{a}c\bar{d}$ could be wrapped around to cover four squares.