SOLUTIONS TO REVIEW PROBLEMS FOR THE FINAL EXAMINATION

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2 .$$

You may leave the solution in implicit form.

Solution: This is a separable ODE:

$$\frac{dy}{(2y+3)^2} = \frac{dx}{(4x+5)^2} \quad \Rightarrow \quad -\frac{1}{2(2y+3)} = -\frac{1}{4(4x+5)} + C, \quad C = \text{constant.}$$

2. Solve the initial value problem

$$x\frac{dy}{dx} + y = e^x , \qquad y(1) = 2$$

Solution: We will solve the ODE for $x > 0$. This is a linear IVP which we'll re-write in the standard form

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$$

to see the integrating factor $\mu(x) = \exp\left(\int \frac{1}{x}\,dx\right) = \exp(\ln x) = x$. Thus

$$xy' + y = e^x \quad \Rightarrow \quad (xy)' = e^x \quad \Rightarrow \quad xy = e^x + c \;\; y = \frac{e^x}{x} + \frac{c}{x}, \quad c = \text{constant.}$$

Since $y(1) = 2$, we have $2 = e + c$, thus $c = 2 - e$, and the solution to the IVP is $y = \frac{e^x}{x} + \frac{2-e}{x}$.

3. Solve the initial value problem

$$x\frac{dy}{dx} + (x-4)y = x^5, \quad y(-1) = 2.$$

Solution: We will solve the ODE for $x < 0$. This is a linear IVP which we'll re-write in the standard form

$$\frac{dy}{dx} + \frac{x-4}{x}y = x^4$$

to see the integrating factor $\mu(x) = \exp\left(\int 1 - \frac{4}{x}\, dx\right) = \exp(x{-}4\ln x) = \exp(x)\exp(\ln(-x)^{-4}) = x^{-4}e^x$. Thus

$$x^{-4}e^x + (x^{-4} - 4x^{-5})e^x y = e^x \quad \Rightarrow \quad (e^x x^{-4} y)' = e^x \quad \Rightarrow \quad e^x x^{-4} y = e^x + c \;\; y = x^4 + cx^4 e^{-x}.$$

Since $y(-1) = 2$, we have $2 = 1 + ce$ and $c = 1/e$, thus the solution to the IVP is

$$y = x^4 + x^4 e^{-x-1}.$$

4. A large tank is filled to capacity with 100 gallons of pure water. Brine containing 3 pounds of salt per gallon is pumped into the tank at a rate of 4 gal/min. The well-mixed solution is pumped out of the tank at the rate of 5 gal/min.

   Find the number of pounds of salt in the tank after 30 minutes.

   Solution: Denote by $A(t)$ the number of pounds of salt in the tank at time t. Then we must solve the IVP

$$\frac{dA}{dt} = 12 - \frac{5A}{100 - t}, \quad A(0) = 0.$$

   The ODE is linear with an integrating factor $\mu(t) = \exp\left(\int \frac{5}{100-t}\, dt\right) = \exp(-5\ln(100{-}t)) = (100-t)^{-5}$.

   So, $(100-t)^{-5}A'(t) + 5A(100-t)^{-4} = 12(100-t)^{-5}$, hence $(100-t)^{-5}A(t) = \frac{12}{4}(100-t)^{-4} + C$ where C will be determined from $A(0) = 0$ to be equal to $-3/100^4$.

   Finally, we have $A(t) = 3(100-t) - \frac{3}{100^4}(100-t)^5$ and $A(30) = 210 - 50.421 = 159.5790$ lb.

5. Find the general solution (explicit or implicit) of the equation

$$(y^2 \cos x - 3x^2 y - 2x)\, dx + (2y \sin x - x^3 + \ln y)\, dy = 0\ .$$

   Solution: It is not hard to check that the equation is exact. Thus, we seek a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = y^2 \cos x - 3x^2 y - 2x)$$

   and

$$\frac{\partial \phi}{\partial y} = 2y \sin x - x^3 + \ln y.$$

   We start by integrating the first equation with respect to x and obtain:

$$\phi(x, y) = y^2 \sin x - x^3 y - x^2 + c(y),$$

which we will now differentiate with respect to y:

$$2y \sin x - x^3 + c'(y) = 2y \sin x - x^3 + \ln y.$$

Thus

$$c(y) = \int \ln y \, dy = \int 1 \cdot \ln y \, dy = y \ln y - \int 1 \, dy = y \ln y - y + k.$$

The general solution of the ODE is then the implicit function $y = y(x)$ given by

$$y^2 \sin x - x^3 y - x^2 + y \ln y - y = C, \quad C = \text{constant}.$$

6. Consider the following linear, second order IVP:

$$(t^2 - 2)y'' + \tan(t)y' + y = \cos(t), \ y(1) = 2, \ y'(1) = 2.$$

What is the maximal open interval of definition of the solution? Motivate your answer.

Solution: Write the ODE in standard form:

$$y'' + \frac{\tan(t)}{(t^2-2)} y' + \frac{1}{(t^2-2)} y = \frac{\cos(t)}{(t^2-2)},$$

and find the largest open interval containing 1 on which all functions $\frac{\tan(t)}{(t^2-2)}, \frac{1}{(t^2-2)}, \frac{\cos(t)}{(t^2-2)}$ are continuous. This interval is $(-\sqrt{2}, \sqrt{2})$.

7. A thermometer is taken from an inside room to the outside, where the air temperature is $5°F$. After 1 minute the thermometer reads $55°F$, and after 5 minutes it reads $30°F$. What is the initial temperature of the inside room?

Solution: The ODE modeling the change in temperature is separable and easy to solve

$$\frac{dT}{dt} = k(T - 5) \quad \Rightarrow \quad T(t) = 5 - Ce^{kt}, \ C = \text{constant}.$$

We will now use that $T(1) = 55, \ T(5) = 30$ to find both C and k. We obtain the equations: $50 = -e^k C, \ 25 = -e^{5k} C$. Divide the second equation by the first and get $e^{4k} = 1/2$, thus $k = \frac{1}{4} \ln(1/2) = \ln(1/2)^{1/4}$. Thus $C = -50 \cdot 2^{1/4}$. Hence

$$T(t) = 5 + 50e^{t \ln 2^{-1/4}} \cdot 2^{1/4} = 5 + 50 \cdot 2^{-t/4} \cdot 2^{1/4} \quad T(0) = 5 + 50 \cdot 2^{1/4} \approx 64.46°F.$$

8. Find a differential equation whose general solution is of the form

$$y = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t).$$

Solution: We are looking first for a characteristic polynomial with the roots: $-2 \pm 3i$, thus $(r - (-2 - 3i))(r - (-2 + 3i)) = r^2 + 4r + 13$. Thus the ODE is

$$y'' + 4y' + 13y = 0.$$

9. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are two solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y(t) = c_1 + c_2 t^{1/2}$ is not, in general, a solution of this equation. Explain why this does not contradict the superposition principle.

   Solution: It is easy to check that each of $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions. Also, that, if neither one of c_1, c_2 is zero, then $y(t) = c_1 + c_2 t^{1/2}$ is not a solution. The superposition principle is not contradicted as the ODE is not linear!

10. Consider the equation $y'' - y' - 2y = 0$.

    (a) Show that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions.

    (b) Let $y_3(t) = -2e^{2t}$, $y_4(t) = y_1 + 2y_2(t)$, and $y_5(t) = 2y_1(t) - 2y_3(t)$. Are y_3, y_4 and y_5 also solutions to the given differential equation? Explain.

    (c) Determine whether each of the following pairs forms a fundamental set of solutions: $\{y_1(t), y_3(t)\}$, $\{y_2(t), y_3(t)\}$, $\{y_1(t), y_4(t)\}$, $\{y_4(t), y_5(t)\}$.

    Solution: (a) It is easy to check that each of them is a solution and that their Wronskian is non-zero for all real t.

    (b) Yes, it is a consequence of the superposition principle. No need to check.

    (c) $\{y_1(t), y_3(t)\}$ yes, as one can check that their Wronskian is nonzero, thus the two functions are linearly independent solutions to the ODE. On the other hand, $\{y_2(t), y_3(t)\}$ do not form a fundamental set as $y_3 = -2y_2$ thus the functions are linearly dependent. The answer is also yes, for the sets $\{y_1(t), y_4(t)\}$ and no for the set $\{y_4(t), y_5(t)\}$ (where $y_5(t) = 2y_4(t)$. Just check that the Wronskian of each of the pairs is non-zero for all t (for a fundamental set) or null otherwise.

11. Find the general solution of the following equation using the method of undetermined coefficients:

$$y'' - y = e^x + e^{-x}.$$

    Solution: The equation $r^2 - 1 = 0$, leads to $y_c(x) = c_1 e^x + c_2 e^{-x}$. Set $y_p(x) = Axe^x + Bxe^{-x}$, then $y_p'(x) = Ae^x + Axe^x + Be^{-x} - Bxe^{-x}$ and $y_p''(x) = 2Ae^x + Axe^x - 2Be^{-x} + Bxe^{-x}$. Thus $y_p'' - y_p = 2Ae^x - 2Be^{-x} = e^x + e^{-x}$ and $A = 1/2,\ B = -1/2$. Hence the general solution:

$$y_{gen}(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{2} xe^x - \frac{1}{2} xe^{-x}, \quad c_{1,2} = \text{constants}.$$

12. Solve the differential equation

$$y'' + y = \sin^2 x.$$

    Solution: The equation $r^2 + 1 = 0$, leads to $y_c(x) = c_1 \cos x + c_2 \sin x$. To find a particular solution, we may change $\sin^2 x = \dfrac{1 - \cos(2x)}{2} = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)$, so we can apply the method of undetermined coefficients where we set $y_p(x) = A + B\cos(2x) + C\sin(2x)$. In this case, we will find $A = 1/2,\ B = 1/6,\ C = 0$.

On the other hand, we can apply directly the method of variation of parameters. First we calculate $W(\cos x, \sin x) = 1$. Then, the formula for the variation of parameters gives:

$$y_p(x) = -\cos x \int \sin x \cdot \sin^2 x\, dx + \sin x \int \cos x \cdot \sin^2 x\, dx$$

$$= -\cos x \int \sin x \cdot (1-\cos^2 x)\, dx + \sin x \int \cos x \cdot \sin^2 x\, dx$$

$$= \cos^2 x - \frac{1}{3}\cos^4 x + \frac{1}{3}\sin^4 x.$$

It may not look obvious that this is the same particular solution obtained with the method of undetermined coefficients. However, note that

$$y_p(x) == \cos^2 x - \frac{1}{3}(1-\sin^2 x)^2 + \frac{1}{3}\sin^4 x = \cos^2 x - \frac{1}{3} + \frac{2}{3}\sin^2 x$$

$$= \frac{\cos 2x + 1}{2} - \frac{1}{3} + \frac{2}{3}\frac{1-\cos 2x}{2} = \frac{1}{2} + \frac{1}{6}\cos(2x).$$

Hence, the general solution is

$$y_{gen}(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6}\cos(2x), \quad c_{1,2} = \text{constants}.$$

13. Find the general solution to

$$2t^2 y'' + ty' - 3y = 0$$

given that t^{-1} is a solution.

Solution: We will start by applying the method of reduction of order to find a second linearly independent) solution to the above homogeneous ODE.
Set $y_2(t) = v(t)t^{-1}$, thus $y_2' = v't^{-1} - vt^{-2}$, and $y_2'' = v''t^{-1} - 2t^{-2}v' + 2vt^{-3}$. So, $2t^2 y_2'' + ty_2' - 3y_2 = 2tv'' - 4v' + v' = 0$, or $2tv'' - 3v' = 0$. Denoting by $u = v'$, we have the separable first order ODE in u: $2tu' - 3u = 0$, thus

$$\frac{u'}{u} = \frac{3}{2}t^{-1} \quad \Rightarrow \quad \ln u = \frac{3}{2}\ln t + c \quad (c = 0) \quad \Rightarrow \quad u = t^{3/2}.$$

Hence $v' = t^{3/2}$ and one v satisfying this ODE is $v = \frac{2}{5}t^{5/2}$, and $y_2(t) = \frac{2}{5}t^{3/2}$.
Thus, the general solution is

$$y_{gen}(t) = c_1 t^{-1} + c_2 t^{3/2}, \quad c_{1,2} = \text{constants}.$$

14. Find the general solution of the differential equation

$$2x^2y'' + 5xy' + y = x^2 - x, \quad x > 0.$$

(Hint: Find solutions of the form x^r to the associated homogeneous equation.)

Solution: Consider first the associated homogeneous equation

$$2x^2y'' + 5xy' + y = 0$$

and follow the hint. Assume that $y(x) = x^r$ is a solution to the homogeneous ODE. Then $y = x^r,\ y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$ satisfy the ODE, so we get the following characterisitc equation:

$$2r(r-1) + 5r + 1 = 0 \quad \Rightarrow \quad 2r^2 + 3r + 1 = 0,$$

with the roots $r = -1$ and $r = -1/2$. Thus $y_1 = \dfrac{1}{x}$ and $y_2(x) = \dfrac{1}{\sqrt{x}}$ are solutions to the homogeneous equation.

Now, put the non-homogeneous equation into the standard form:

$$y'' + \frac{5}{2x}\,y' + \frac{1}{2x^2}\,y = \frac{1}{2} - \frac{1}{2x}.$$

You can now apply the method of variation of parameters for y_1, y_2 and $g(x) = \dfrac{1}{2} - \dfrac{1}{2x}$.

We calculate the Wronskian of y_1, y_2, and the matrices W_1, W_2:

$$W = \det\begin{pmatrix} x^{-1} & x^{-1/2} \\ -x^{-2} & -\frac{1}{2}x^{-3/2} \end{pmatrix} = \frac{1}{2x^{5/2}}, \quad W_1 = \det\begin{pmatrix} 0 & x^{-1/2} \\ 1 & -\frac{1}{2}x^{-3/2} \end{pmatrix} = -\frac{1}{\sqrt{x}}, \quad W_2 = \det\begin{pmatrix} x^{-1} & 0 \\ -x^{-2} & 1 \end{pmatrix} = \frac{1}{x}.$$

Hence, a particular solution to the non-homogeneous ODE is:

$$y_p(x) = \frac{1}{x}\int\left(1 - \frac{1}{x}\right)x^2\,dx + \frac{1}{\sqrt{x}}\int\left(1 - \frac{1}{x}\right)x^{3/2}dx = -\frac{1}{x}\int\left(x^2 - x\right)\,dx + \frac{1}{\sqrt{x}}\int\left(x^{3/2} - x^{1/2}\right)\,dx$$

and taking the constants of integration to be zero:

$$y_p(x) = -\frac{x^2}{3} + \frac{x}{2} + \frac{2}{5}x^2 - \frac{2}{3}x = \frac{1}{15}x^2 - \frac{1}{6}x.$$

Thus, the general solution of the original ODE is:

$$y(x) = c_1\frac{1}{x} + c_2\frac{1}{\sqrt{x}} + \frac{1}{15}x^2 - \frac{1}{6}x,$$

where $c_{1,2}$ are arbitrary constants.

15. Determine the general solution of the equations

$$(a) \quad y^{(4)} - 4y'' = t^2 + e^t.$$

$$(b) \quad y''' - 2y'' - y' + 2y = e^{-t}\sin t.$$

Solution: (a) Since $r^4 - 4r^2 = 0$, the roots are $0, 0, \pm 2$, hence $y_c(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t}$. Consider $y_p(t) = t^2(At^2 + Bt + C) + De^t$, $y_p'(t) = 4At^3 + 3Bt^2 + 2Ct + De^t$, $y_p''(t) = 12At^2 + 6Bt + 2C + De^t$, $y_p'''(t) = 24At + 6B + De^t$, $y_p^{(4)}(t) = 24A + De^t$, thus

$$24A + De^t - 4(12At^2 + 6Bt + 2C + De^t) = t^2 + e^t \;\Rightarrow\; D = -1/3,\; A = -1/48,\; B = 0,\; C = 3A = -1/6.$$

Thus

$$y_p(t) = -\frac{1}{48}\,t^4 - \frac{1}{16}\,t^2 - \frac{1}{3}\,e^t$$

and

$$y_{gen}(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} - \frac{1}{48}\,t^4 - \frac{1}{16}\,t^2 - \frac{1}{3}\,e^t, \quad c_{1,2,3,4} = \text{constants}.$$

(b) We have $r^3 - 2r^2 - r + 2 = 0 \;\Rightarrow\; (r-2)(r^2-1) = 0$, thus $r = \pm 1, 2$. Hence

$$y_c(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}.$$

On the other hand, $y_p(t) = e^{-t}(A\sin t + B\cos t)$, $y + p'(t) = e^{-t}(A\cos t - B\sin t) - e^{-t}(A\sin t + B\cos t)$, $y_p''(t) = -2e^{-t}(A\cos t - B\sin t)$, $y_p'''(t) = -2e^{-t}(-A\sin t - B\cos t) + 2e^{-t}(A\cos t - B\sin t)$.

Hence

$$y_p''' - 2y_p'' - y_p' + 2y_p = e^{-t}\left[\sin t(5A - 5B) + \cos t(5A + 5B)\right] \;\Rightarrow\; A = \frac{1}{10}, \quad B = -\frac{1}{10}$$

and

$$y_{gen}(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + \frac{e^{-t}}{10}(\sin t - \cos t), \quad c_{1,2,3} = \text{constants}.$$

16. Given the LRC-circuit with $L = \dfrac{5}{3}$ henries, $R = 10$ ohms, $C = \dfrac{1}{3}$ farads, and $E(t) = 50\cos t$ volts, the charge $q(t)$ satisfies the linear second order ordinary differential equation

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = E(t).$$

(a) Find the charge $q(t)$ if $q(0) = 100$ coulombs and $q'(0) = 0$ amperes.

(b) Identify in $q(t)$ the transient terms and, respectively, the steady state terms. Is the circuit described by $L\,\dfrac{d^2q}{dt^2}+R\,\dfrac{dq}{dt}+\dfrac{1}{C}\,q=0$ overdamped, underdamped, or critically damped?

Solution: We need to use the ODE:

$$\frac{5}{3}q''+10q'+3q=50\cos t.$$

(a) Consider first the homogeneous equation which will be solved via the characteristic equation to obtain:

$$q_{gen}^{hom}(t)=c_1e^{\frac{-15-\sqrt{180}}{10}t}+c_2e^{\frac{-15+\sqrt{180}}{10}t},\quad c_{1,2}=\text{constants}.$$

To find a particular solution, you may use the method of undetermined coefficients, setting $q_p(t)=A\cos t+B\sin t$. Doing the usual calculations, we obtain: $A=\dfrac{600}{916}$, $B=\dfrac{4500}{916}$.
Thus, the general solution of the initial ODE:

$$q_{gen}^{non-hom}(t)=c_1e^{\frac{-15-\sqrt{180}}{10}t}+c_2e^{\frac{-15+\sqrt{180}}{10}t}+\frac{600}{916}\cos t+\frac{4500}{916}\sin t,\quad c_{1,2}=\text{constants}.$$

We will now use the initial conditions to find the constants:

$$c_2=\frac{91000}{916}\left(-\frac{15}{2\sqrt{180}}+\frac{1}{2}\right),\quad c_1=\frac{91000}{916}\left(\frac{15}{2\sqrt{180}}+\frac{1}{2}\right).$$

(b) The circuit is overdamped. (See the definition in the textbook.) The transient term is:

$$c_1e^{\frac{-15-\sqrt{180}}{10}t}+c_2e^{\frac{-15+\sqrt{180}}{10}t},$$

while the steady-state term is

$$\frac{600}{916}\cos t+\frac{4500}{916}\sin t.$$

17. A force of 200 newtons stretches a spring 1 meter. A mass of 400 kilograms is attached to the end of the spring such that its displacement $x(t)$ satisfies the equation

$$400\,\frac{dx^2}{dt^2}+200\,x=400\cos(t),\quad x(0)=0\text{ m},\quad x'(0)=-10\text{ m/s}.$$

Give the physical interpretation of the ordinary differential equation for the mass-spring system. Interpret also the initial conditions.

Solution: This models a mass-spring system not subjected to damping but to an external force of $F(t)=400\cos t$ N which starts at equilibrium with an upward speed of 10 m/s.

Note that $x_c(t) = c_1 \cos(t\sqrt{2}/2) + c_2 \sin(t\sqrt{2}/2)$ and choose $x_p(t) = A\cos t + B\sin t$. Then $2x'' + x = -A\cos t - B\sin t = 2\cos t$, hence $A = -2,\ B = 0$.

Thus

$$x_{gen}(t) = c_1 \cos(t\sqrt{2}/2) + c_2 \sin(t\sqrt{2}/2) - 2\cos t$$

and $x(0) = 0$ implies $c_1 = 2$, while $x'(0) = -10$ implies $c_2 = -10/sqrt2$. Thus

$$x_{gen}(t) = 2\cos(t\sqrt{2}/2) - 10\sqrt{2}\,\sin(t\sqrt{2}/2) - 2\cos t.$$

18. Use the Laplace transform to solve each of the following initial value problems

$$(a) \quad y'' + 2y' + 5y = 0,\ y(0) = 2,\ y'(0) = -1$$

and

$$(b) \quad y^{(4)} - 4y''' + 6y'' - 4y' + y = 0,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = -2,\ y'''(0) = 0.$$

Solution: (a) $\mathcal{L}\{y'' + 2y' + 5y\} = 0 \quad \Rightarrow \quad s^2\mathcal{L}\{y\} - 2s + 1 + 2(s\mathcal{L}\{y\} - 2) + 5\mathcal{L}\{y\} = 0$. Thus $\mathcal{L}\{y\} = \dfrac{2s+3}{s^2+2s+5} = 2\dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{(s+1)^2+4}$. From here, taking the Laplace inverse, we obtain

$$y(t) = 2e^{-t}\cos(2t) + \frac{1}{2}\,e^{-t}\sin(2t).$$

(b)$\mathcal{L}\{y^{(4)} - 4y''' + 6y'' - 4y' + y\} = 0 \quad \Rightarrow \quad s^4\mathcal{L}\{y\} - s^2 + 2s - 4(s^3\mathcal{L}\{y\} - s + 2) + 6(s^2\mathcal{L}\{y\} - 1) - 4s\mathcal{L}\{y\} + \mathcal{L}\{y\} = 0$.

Thus

$$\mathcal{L}\{y\}(s^4 - 4s^3 + 6s^2 - 4s + 1) = s^2 - 6s + 14 \quad \Longrightarrow \quad \mathcal{L}\{y\} = \frac{s^2 - 6s + 14}{s^4 - 4s^3 + 6s^2 - 4s + 1}.$$

Now, since

$$\frac{s^2 - 6s + 14}{s^4 - 4s^3 + 6s^2 - 4s + 1} = \frac{s^2 - 6s + 14}{(s-1)(s^3 - 3s^2 + 3s - 1)}$$

$$= \frac{s^2 - 6s + 14}{(s-1)(s^3 - 3s^2 + 3s - 1)} = \frac{s^2 - 6s + 14}{(s-1)(s-1)^3} = \frac{s^2 - 6s + 14}{(s-1)^4},$$

we look for its partial fractions decomposition:

$$\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4} = \frac{s^2 - 6s + 14}{(s-1)^4}.$$

We will get $A = 0,\ B = 1,\ C = -4,\ D = 9$.

Thus, using formula 11 repeatedly,

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{(s-1)^2} - \frac{4}{(s-1)^3} + \frac{9}{(s-1)^4}\} = te^t - 2t^2e^t + \frac{3}{2}t^3e^t.$$

19. Find the inverse Laplace transform of

$$F(s) = \frac{2(s-1)e^{-2s}}{s^2 - 2s + 2}.$$

Solution: The inverse Laplace transform of $\dfrac{2(s-1)}{s^2-2s+2} = 2\dfrac{s-1}{(s-1)^2+1}$ is, by formula 10, $2e^t \cos t$. We can now apply formula number 13, to conclude that the inverse Laplace transform of $F(s)$ is

$$2u_2(t)e^{t-2}\cos(t-2).$$

20. Apply the convolution theorem to find the inverse Laplace transform of the function

$$F(s) = \frac{1}{s(s-3)}.$$

Solution: The inverse Laplace transform of $\frac{1}{s}$ is the function 1 and the inverse Laplace transform of $\frac{1}{s-3}$ is the function e^{3t}, thus, by convolution theorem, the inverse Laplace transform of F is the convolution:

$$1 \star e^{3t} = \int_0^t e^{3\tau} d\tau = \frac{e^{3\tau}}{3} \Big|_0^t = \frac{e^{3t}}{3} - \frac{1}{3}.$$