Multivariable calculus (MATH-234/MAST218) Winter 2014

Midterm test solutions

Problem 1. Find the length of the curve $x=15t^{7}, \; y=42t^{5}, \; z=70t^{3}$, $0\le t\le 2$.

Solution. The length is defined by the formula

$$L=\int_0^2 \sqrt{\dot{x}^2(t)+\dot{y}^2(t)+\dot{z}^2(t)}\,dt$$

In our problem, $\dot{x}(t)=105t^{6}, \; \dot{y}(t)=210t^{4}, \; \dot{z}(t)=210t^{2}$. So,

$$L=\int_0^2 \sqrt{(105t^6)^2+(210t^4)^2+(210t^2)^2}\,dt$$

$$=\int_0^2 105\sqrt{t^{12}+4t^8+4t^4}\,dt=\int_0^2 105\sqrt{(t^4+2)^2}\,dt$$

$$=\int_0^2 105t^2(t^4+2)\,dt=\int_0^2 (105t^6+210t^2)\,dt$$

$$=105\left[\frac{2^7}{7}\right]+210\left[\frac{2^3}{3}\right]=2480.$$

Problem 2. (a) Find the equation of a parabola with the directrix $x=3$ and the focus $(-1,-2)$.

(b) Find the vertex of the parabola you've found and the points of its intersection with the coordinate axes.

Solution. Parabola is defined as a locus of points which are equidistant from the directrix and the focus. If (x, y) are coordinates of a point of parabola, then its distance from the directrix is $3-x$, and its distance to the focus is $\sqrt{(x+1)^2+(y+2)^2}$. So, the equation of parabola is

$\sqrt{(x+1)^2+(y+2)^2}=3-x;$ squaring the left and the right hand sides, we get the equation $(x+1)^2+(y+2)^2=(3-x)^2$, or, after some simplification, $(y+2)^2=8(1-x)$. The vertex is $(-1,-2)$. To find the intersection with the x axis, we set $y=0$, and get equation $8(1-x)=4$; $x=\frac{1}{2}$. So, the x intercept is $\left(\frac{1}{2,}0\right)$.

To find the y intercept, we set $x=0$, and thus we have to solve the equation $(y+2)^2=8;$ $y+2=\pm 2\sqrt{2};$ $y=-2\pm 2\sqrt{2}$. So, the y intercepts are the points $(0,2-2\sqrt{2})$ and $(0,2+2\sqrt{2})$.

Problem 3. Find the equation of a plane which is orthogonal to the planes $P1:\ x+2y+3z=1$ and $P2:\ 3x+2y+z=1$.

Solution. The plane $P1$ is orthogonal to the vector $\boldsymbol{u}=(1,2,3)$, and the plane $P2$ is orthogonal to the vector $\boldsymbol{v}=(3,2,1)$. If $\boldsymbol{w}=\boldsymbol{u}\times\boldsymbol{v}$, then the vector $\boldsymbol{w}$ is orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$, and hence a plane orthogonal to the vector $\boldsymbol{w}$ is parallel to the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, and therefore is orthogonal to the planes $P1$ and $P2$. So, we have to find the vector $\boldsymbol{w}$.

$$\boldsymbol{w}=\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & 3\\ 3 & 2 & 1\end{vmatrix}=(2-6)\boldsymbol{i}+(9-1)\boldsymbol{j}+(2-6)\boldsymbol{k}=(-4,8,-4).$$

Equation of the plane orthogonal to the vector $\boldsymbol{w}$ and containing the point $(1,2,3)$ is

$-4(x-1)+8(y-2)-4(z-3)=0$, or

$-4x+8y-4z=0$, or $x-2y+z=0$.

Problem 4. Find the angle between the curves $x=t,\ y=t^{2,}\ z=t^3$ and $x=\tan t,\ y=\cot t,\ z=\tan^2 t$ at the point (1,1,1).

Solution. Let us write the vector equations of the curves:

$$\boldsymbol{r}_1(t)=(t, t^{2,} t^3)\ ;\quad \boldsymbol{r}_2(t)=(\tan t, \cot t, \tan^2 t)\ .$$

Then $\boldsymbol{r}_1(t)=(1,1,1)$ for $t=1$ while $\boldsymbol{r}_2(t)=(1,1,1)$ for $t=\frac{\pi}{4}$. The tangent (velocity) vectors are $\dot{\boldsymbol{r}}_1(t)=(1,2t,3t^2)$ and

$$\dot{\boldsymbol{r}}_2(t)=(\sec^2 t, -\operatorname{cosec}^2 t, 2\tan t\sec^2 t)=\left(\frac{1}{\cos^2 t}, -\frac{1}{\sin^2 t}, \frac{2\tan t}{\cos^2 t}\right)\ .$$

So, $\boldsymbol{u}=\dot{\boldsymbol{r}}_1(1)=(1,2,3)$, and $\boldsymbol{v}=\dot{\boldsymbol{r}}_2\left(\frac{\pi}{4}\right)=(2,-2,4)$. The angle α between the curves at their intersection point (1,1,1) is, by definition, equal to the angle between the tangent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\cos\alpha=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}|\cdot|\boldsymbol{v}|}=\frac{10}{\sqrt{14}\sqrt{24}}=0.5455\ .\ \text{Then}\ \ \alpha=\arccos\frac{10}{\sqrt{14}\sqrt{24}}=0.9937\ .$$

Problem 5. Find the curvature of the graph of the function $y=x^3$ at the point (1,1).

Solution. The parametric vector equation of the graph is $\boldsymbol{r}(t)=(t, t^{3,} 0)$. Then $\dot{\boldsymbol{r}}(t)=(1,3t^2,0)$, $\ddot{\boldsymbol{r}}(t)=(0,6t,0)$; in particular, $\boldsymbol{r}(1)=(1,1,0)$, $\dot{\boldsymbol{r}}(1)=(1,3,0)$, $\ddot{\boldsymbol{r}}(1)=(0,6,0)$. The curvature κ is defined by the formula

$\kappa = \dfrac{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|}{|\dot{\boldsymbol{r}}|^3}$.

$\dot{\boldsymbol{r}}(1) \times \ddot{\boldsymbol{r}}(1) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 0 \\ 0 & 6 & 0 \end{vmatrix} = 6\boldsymbol{k}$; so, $|\dot{\boldsymbol{r}}(1) \times \ddot{\boldsymbol{r}}(1)| = 6$. Next,

$|\dot{\boldsymbol{r}}(1)| = \sqrt{1^1 + 3^2} = \sqrt{10}$; hence, $\kappa = \dfrac{6}{10\sqrt{10}}$.