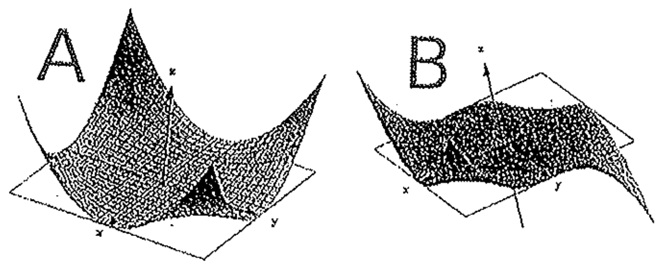
Sample Midterm 1 for Math 105

1. Let $z = f(x, y) = x^2y^2$.

(a) Sketch the level curves $f(x, y) = z_0$ with $z_0 = 0$ and $z_0 = 1$.

(b) Which of the following two surfaces correspond to the graph of f? Decide and justify your answer!



(c) Find an equation of the plane passing through a point $P(2, -3, f(2, 3))$ with a normal vector $\vec{n} = < -1, 3, 2 >$.

(d) Does the equation $2x - 6y - 4z = -122$ describes the same plane in (c)? Justify your answer.

2. Find *all* critical points of the function

$$f(x, y) = 3x^2 - 6xy + y^3 - 9y.$$

Classify each point as a local maximum, local minimum, or saddle point.

3. A company wishes to build a new warehouse. It should be situated on the northeast quarter of the Oval, an expressway whose shape is given by the equation

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

Here x and y are measured in kilometers. From the company's point of view, the desirability of a location on the Oval is measured by the sum of its horizontal and vertical distances from the origin. The larger the sum is, the more desirable the location is. Find using the method of Lagrange multipliers the location on the Oval that is most desirable to the company.

Clearly state the objective function and the constraint. There is no need to justify that the solution you obtained is the absolute max or min. **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.**

4. (a) Let $f(x,y) = \ln(9 - x^2 - y^2)$. Sketch the domain of f in the xy-plane and find f_{xx}.

(b) Show or disprove that there exist a function g which has continuous partial derivatives of all orders such that

$$g_x = 9998x^{9998}y \quad \text{and} \quad g_y = x^{9999}.$$

5. Let R be the semicircular region $\{x^2 + y^2 \le 9,\ y \ge 0\}$. Find the maximum and minimum values of the function

$$f(x,y) = x^2 + y^2 - 4y$$

on the *boundary of the region* R.

6. A function $f(x)$ is defined on the interval $[-2,4]$ as follows:

$$f(-2) = f(2) = f(4) = 0, \quad f(0) = 4, \quad f(3) = -1,$$

and the graph of f consists of straight line segments joining these points. Compute the value of the integral

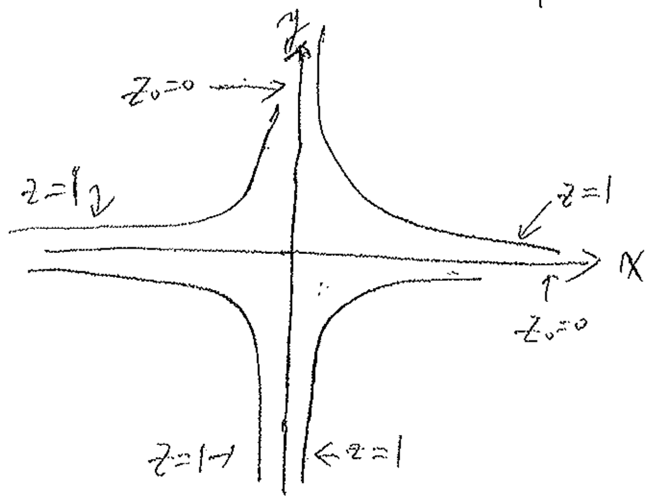
$$\int_{-2}^{4} f(x)\,dx.$$

7. Transform the limit of the following Riemann sum to a definite integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1 + \left(\frac{k}{n}\right)^2}.$$

Solutions to Sample Midterm 1 for Math 105

1. (a)



The level curve $f(x,y)=0$ consists of the x-axis and the y-axis, and the level curve $f(x,y)=1$ consists of the graph of $y=\frac{1}{x}$ and the graph of $y=-\frac{1}{x}$.

(b) Since $f(x,y)\geq 0$ for all $(x,y)\in\mathbb{R}^2$, the surface B is not the graph of f. Hence, the surface A is the graph of f.

(c) Since $f(2,3)=2^2\cdot 3^2=36$, an equation of the plane is given by

$$-1(x-2)+3(y+3)+2(z-36)=0$$

or $$-x+3y+2z=61.$$

(d) Yes. because the normal vector given by the equation in (d) is $-2\vec{n}$, and $P(2,-3,f(2,3))$

is contained in the plane given by the equation in (d). //

2. $\begin{cases} f_x = 6x - 6y = 0 \\ f_y = -6x + 3y^2 - 9 = 0 \end{cases} \Rightarrow (x, y) = (3, 3)$ or $(-1, -1)$

$f_{xx} = 6, \ f_{xy} = -6, \ f_{yy} = 6y$

Critical points	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	Conclusions
$(3, 3)$	6	18	-6	$+$	local min.
$(-1, -1)$	6	-6	-6	$-$	saddle point

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3. Maximize $f(x, y) = x + y$
subject to $g(x, y) = \frac{x^2}{9} + \frac{y^2}{16} - 1 = 0$, $x \geq 0, y \geq 0$

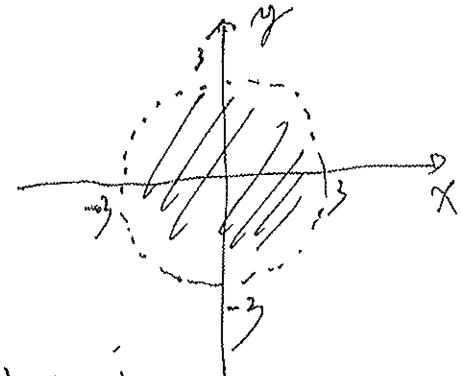
$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Rightarrow \begin{cases} 1 = \lambda \cdot \frac{2x}{9} & (1) \\ 1 = \lambda \cdot \frac{2y}{16} & (2) \\ \frac{x^2}{9} + \frac{y^2}{16} = 1 & (3) \end{cases}$

(1) and (2) $\Rightarrow \frac{2x}{9} = \frac{2y}{16}$

$\Rightarrow y = \frac{16}{9} x.$ (4)

By (4), (3) becomes $\frac{x^2}{9} + \frac{1}{16} \cdot \frac{16^2}{9^2} x^2 = 1 \Rightarrow x = \frac{9}{5}$ and $y = \frac{16}{5}$. //

4. (a) The domain of f is $\{(x,y): x^2+y^2<9\}$, which consists of all points in $\mathbb{R}^2$ which are inside the circle: $x^2+y^2=9$.



Since $f_x = \dfrac{-2x}{9-x^2-y^2}$, we get

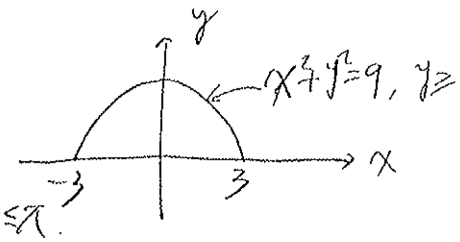
$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{-2x}{9-x^2-y^2}\right) = \frac{-2(9-x^2-y^2)-(-2x)(-2x)}{(9-x^2-y^2)^2}$$

or $f_{xx} = \dfrac{-18-2x^2+2y^2}{(9-x^2-y^2)^2}$.

(b) Theres is no function g with the property because we must have $g_{xy} = g_{yx}$. However, in our case,

$$g_{xy} = \frac{\partial}{\partial y}(g_x) = 9998x^{9998} \neq 9999x^{9998} = g_{yx}.$$

5. On the semicircle part of



the boundary, $x=3\cos\theta$, $y=3\sin\theta$, $0\le\theta\le\pi$.

Hence, $g(\theta)=f(3\cos\theta, 3\sin\theta)=9-12\sin\theta$, $0\le\theta\le\pi$

$g'(\theta)=-12\cos\theta \overset{\text{set}}{=} 0 \Rightarrow \theta=\frac{\pi}{2}$. $\boxed{g(\frac{\pi}{2})=9-12=-3}$, $\boxed{g(0)=g(\pi)=9}$.

On the line segment part of the boundary, $y=0$ and $-3\le x\le 3$

$f(x,0)=x^2$, $-3\le x\le 3$, which has max. value $\boxed{f(\pm 3,0)=9}$ and min. value $\boxed{f(0,0)=0}$

By the two parts above, 9 is the max. value of f on the boundary of R, and -3 is the min. value of f on the boundary of R.

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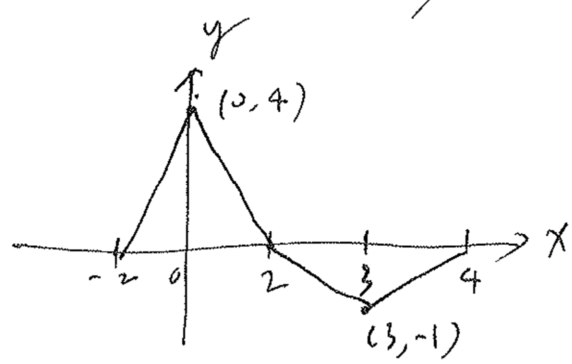
6. $\int_{-2}^{4} f(x)\,dx=\int_{-2}^{2} f\,dx+\int_{2}^{4} f\,dx$

$=\frac{1}{2}\cdot 4\cdot 4-\frac{1}{2}\cdot 2\cdot 1=8-1=7.$



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7. $\int_0^1 \frac{1}{1+x^2}\,dx.$

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