PMATH 340 **Assignment 5, due April 4**

1. Let α be the real number given by the infinite continued fraction with partial quotients all equal to 1, i.e. $q_0 = 1, q_1 = 1, q_2 = 1, \ldots$. Let F_1, F_2, F_3, etc denote the Fibonacci numbers $1, 1, 2, 3, 5, \ldots$. Recall that the Fibonacci sequence is defined by $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.
a) Show that the convergents A_m/B_m to α are given by $A_m = F_{m+2}$, $B_m = F_{m+1}$.
b) Show that $\alpha = (1 + \sqrt{5})/2$.
(This proves that the ratio of two consecutive Fibonacci numbers, F_{m+1}/F_m, tends to the golden ratio).

2. Let n be a positive integer. Show that the continued fraction for $\sqrt{2 + n^2}$ has partial quotients $n, n, 2n, n, 2n, n, 2n, \ldots$, i.e. the first quotient is n, and the subsequent quotients are, alternately, n and $2n$.

3. Let α be an irrational number whose continued fraction has partial quotients $q_0, q_1, q_2, \ldots$, and convergents $A_0/B_0, A_1/B_1, A_2/B_2, \ldots$. Show that

$$\alpha = q_0 + \sum_{m=0}^{\infty} \frac{(-1)^m}{B_m B_{m+1}}.$$

Hint: consider $\sum (A_{m+1}/B_{m+1} - A_m/B_m)$.

4. Let $2n$ be a positive even integer, with $n \geq 3$. Determine the continued fraction expansion of $(2n)^2/(2n - 1)^2$.