

221-202 Final Exam - Review Sheet

1. TRUE or FALSE

(a) If 0 is an eigenvalue of  $A$ , then  $A$  is not invertible. *true*

(b) The eigenspace of an eigenvalue  $\lambda$  of a matrix  $A$  consists of all the eigenvectors corresponding to  $\lambda$ .

*false* The eigenspace also contains the zero vector, which is not an eigenvector.

(c) Suppose that  $A$  is a  $222 \times 222$  matrix. Then  $\det(-A) = \det(A)$ . *true*

$$\det(-A) = (-1)^{222} \det A = \det A$$

(d) If 1 is the only eigenvalue of  $A$ , then  $A$  must be the identity matrix. *false*

e.g.  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(e) If  $\vec{v}$  and  $\vec{w}$  are linearly independent eigenvectors of the matrix  $A$ , then  $\vec{v} + \vec{w}$  must also be an eigenvector of  $A$ . *false* If  $\vec{v}$  and  $\vec{w}$  have different eig. values, then  $\vec{v} + \vec{w}$  is not an eig. vector

(f) If an  $n \times n$  matrix  $A$  is diagonalizable, then every vector  $\vec{v}$  in  $\mathbb{R}^n$  can be expressed as a linear combination of eigenvectors of  $A$ . *true*

(g) If  $A$  is a square matrix such that  $A^T A = A A^T$ , then  $\text{Null}(A)$  must equal  $\text{Null}(A^T)$ . *true*

*Key fact:  $\text{Null}(A) = (\text{Col}(A^T))^\perp$ . Now  $\vec{v} \in \text{Null}(A) \Rightarrow A\vec{v} = \vec{0} \Rightarrow A^T A\vec{v} = \vec{0} \Rightarrow A A^T \vec{v} = \vec{0} \Rightarrow A^T \vec{v} \in \text{Null}(A) = (\text{Col}(A^T))^\perp$ . Therefore  $A^T \vec{v}$  is in both  $\text{Col}(A^T)$  and its orthog complement  $\Rightarrow A^T \vec{v} = \vec{0} \Rightarrow \vec{v} \in \text{Null}(A^T)$ .*

(h) The dot product  $\vec{v} \cdot \vec{w} = 0$  if and only if at least one of  $\vec{v}$  or  $\vec{w}$  is the zero vector. *false*

e.g.  $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(i) If  $W$  is a linear subspace of  $\mathbb{R}^n$ , then  $W$  and  $W^\perp$  have no vectors in common. *false*

*The zero vector is in both  $W$  and  $W^\perp$ .*

(j) If a vector  $\vec{x}$  is orthogonal to vectors  $\vec{v}$  and  $\vec{v} + \vec{w}$ , then it is orthogonal to  $\vec{w}$ . *true*

$$\vec{x} \cdot (\vec{v} + \vec{w}) = \vec{x} \cdot \vec{v} + \vec{x} \cdot \vec{w} = 0 + 0 = 0.$$

(k) If  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is an orthogonal set of vectors, and if  $c_1, c_2, c_3 \in \mathbb{R}^n$ , then  $\{c_1 \vec{v}_1, c_2 \vec{v}_2, c_3 \vec{v}_3\}$  is an orthogonal set.

$$(c_i \vec{v}_i) \cdot (c_j \vec{v}_j) = c_i c_j \vec{v}_i \cdot \vec{v}_j = 0$$

*true*

(l) If a square matrix has orthogonal columns, then it also has orthogonal rows. *false*

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

*columns are orthogonal, rows are not.*

2. Consider the matrix  $A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{pmatrix}$ . Show that 1 is an eigenvalue of  $A$ . Find the characteristic polynomial of  $A$  and the remaining eigenvalues of  $A$ . What is the algebraic multiplicity of each eigenvalue? Is  $A$  diagonalizable? If not, why not? If so, find a diagonal matrix  $D$  and an invertible matrix  $P$  so that  $A = P^{-1}DP$ .

$$p_A(\lambda) = (2-\lambda)^2(1-\lambda)$$

The eig. val's of  $A$  are: 1 with alg. multiplicity 1  
2 with alg. mult. 2.

$$A - 2I = \begin{pmatrix} 1 & -2 & 5 \\ 1 & -2 & 7 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(E_2) = 1$$

Therefore  $A$  is Not diagonalizable.

3. Suppose the weather in a certain city is classified as either rainy or dry. Based on extensive record-keeping, the probability of a rainy day following a rainy day is 90%, and the probability of a rainy day following a dry day is ~~40%~~ **40%**.

Let  $\vec{w}_n = \begin{pmatrix} r_n \\ d_n \end{pmatrix}$  be such that  $r_n$  is the probability (expressed as a decimal number  $\leq 1$ ) that the  $n$ th day of 2014 is rainy, and  $d_n$  is the probability that the  $n$  day of 2014 is dry. Based on the data, find a  $2 \times 2$  matrix  $A$  so that  $\vec{w}_{n+1} = A\vec{w}_n$ . Suppose that January 1, 2014, is a rainy day; then what is the probability that January 3, 2014, is rainy? What is the approximate probability that January 1, 2016, is a rainy day? Or, in mathematical terms, what is the state of equilibrium for this system? (Your solution should involve an eigenvector with its interpretation.)

$$A = \begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}$$

January 1, 2014 is day 0:  $\vec{w}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow \text{rainy}$

$$\text{January 3 } \vec{w}_3 = A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.85 & 0.6 \\ 0.15 & 0.4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.85 \\ 0.15 \end{pmatrix} \Rightarrow \text{There is an 85\% probability of rain on January 3, 2014.}$$

$$p_A(\lambda) = (0.5 - \lambda)(1 - \lambda)$$

The steady state corresponds to the eig. vector of the eig. val. 1 whose entries represent probabilities.

$$A - I = \begin{pmatrix} -0.1 & 0.4 \\ 0.1 & -0.4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 4x_2 & x_1 + x_2 = 1 \\ \Rightarrow x_1 = 0.8 & x_2 = 0.2 \end{matrix} \quad \leftarrow \begin{matrix} \text{since } x_1, x_2 \\ \text{are probabilities} \end{matrix}$$

Therefore there is an 80% probability that Jan 1, 2016 is rainy.

4. Find a basis for  $(\text{Row}(A))^{\perp}$ , where  $A$  is the matrix  $\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$ .

$$(\text{Row}(A))^{\perp} = \text{Nul}(A)$$

$$A \sim \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{a basis is } \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_3 = -2x_4 + 2x_5$$

5. Let  $W$  be the linear subspace spanned by the vectors  $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ . Find the matrix for orthogonal projection onto  $W$ . Find a nonzero vector which is orthogonal to  $W$ .

Let  $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$  The matrix of orthogonal projection onto  $W$  is  $A(A^T A)^{-1} A^T$

$$A^T A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow (A^T A)^{-1} = \frac{1}{3} I$$

$$A \left( \frac{1}{3} I \right) A^T = \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

A vector orthogonal to  $W$  is any vector in  $\text{Null}(A^T)$ , e.g.,  $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ .

6. Find a least squares solution to  $A\vec{x} = \vec{b}$ , where  $A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}$ .

$\vec{x}^*$  least square solution

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} \quad A^T A = \begin{pmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & 42 \end{pmatrix}$$

$$(A^T A)^{-1} A^T = \begin{pmatrix} 1/14 & 0 \\ 0 & 1/42 \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1/14 & 3/14 & -2/14 \\ 5/42 & 1/42 & 4/42 \end{pmatrix}$$

$$\vec{x}^* = \begin{pmatrix} 1/14 & 3/14 & -2/14 \\ 5/42 & 1/42 & 4/42 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{5-6+6}{14} \\ \frac{25-2-12}{42} \end{pmatrix} = \begin{pmatrix} 5/14 \\ -11/42 \end{pmatrix}$$

7. Find a linear function  $p(t) = a + bt$  with  $a, b \in \mathbb{R}$  that best fits the data  $(2, 3), (3, 2), (5, 1), (6, 0)$ .

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \quad A\vec{x}^* = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \quad \vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 16 \\ 16 & 74 \end{pmatrix} \quad (A^T A)^{-1} = \frac{1}{40} \begin{pmatrix} 4 & 16 \\ 16 & 74 \end{pmatrix}$$

$$= \begin{pmatrix} 1.85 & -0.4 \\ -0.4 & 0.1 \end{pmatrix}$$

$$(A^T A)^{-1} A^T \vec{b} = \begin{pmatrix} 4.3 \\ -0.7 \end{pmatrix} \quad p(t) = 4.3 - 0.7t$$

8. Suppose that the matrix of a dynamical system  $\vec{x}_{n+1} = A\vec{x}_n$  is

$$A = \begin{pmatrix} 0.5 & 0.6 \\ -0.3 & 1.4 \end{pmatrix}$$

Classify the origin as an attractor, a repeller, or a saddle point. Find a closed form for  $\vec{x}_n$  if  $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

$$\det(A - \lambda I) = (1.1 - \lambda)(0.8 - \lambda)$$

$1.1 > 1$ ,  $0.8 < 1 \Rightarrow$  the origin is a saddle point.

$$A - 1.1I = \begin{pmatrix} -0.6 & 0.6 \\ -0.3 & 0.3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ eig. vector}$$

$$\vec{x}_n = (1.1)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

9. The amounts for two types of cereal, Cereal X and Cereal Y, of calories, protein, carbohydrate, and fat, are listed in the table below. Suppose we want a mixture of the two cereals that contains exactly 295 calories, 9 g of protein, 48 g of carbohydrate, and 8 g of fat. Set up a vector equation for this problem, keeping track of what your variables represent. Express this as a matrix-vector equation, and then determine if the desired mixture of the two cereals can be prepared.

Nutrient	Cereal X	Cereal Y
Calories	110	130
Protein	4	3
Carbohydrate	20	18
Fat	2	5

*C = # of calories   P = # of grams of protein   R = # of g of carbo   F = grams of fat*

$$\begin{pmatrix} 110 & 130 \\ 4 & 3 \\ 20 & 18 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 295 \\ 9 \\ 48 \\ 8 \end{pmatrix} \quad \begin{array}{l} X = \text{amount of Cereal X} \\ Y = \text{amount of Cereal Y} \end{array}$$

*augmented matrix:*

$$\left( \begin{array}{cc|c} 110 & 130 & 295 \\ 4 & 3 & 9 \\ 20 & 18 & 48 \\ 2 & 5 & 8 \end{array} \right)$$

$$X = \frac{3}{2} \quad Y = 1$$

10. Consider the plane  $2x + y - z = 0$ . Find a nonzero vector which is orthogonal to this plane. Suppose that  $P$  is the matrix representing orthogonal projection onto this plane. What is the nullspace of  $P$ ?

$$\begin{aligned} \text{The plane} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x + y - z = 0 \right\} & \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x + y - z \\ & & \Rightarrow \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ is orthogonal to the plane} \end{aligned}$$

The nullspace of  $P$  consists of the set of all vectors orthogonal to the plane. Since the plane is in  $\mathbb{R}^3$ , it is one-dimensional. Since it contains  $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ ,  $\text{Null}(P) = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}$ .

11. Suppose that  $W$  is an  $m$  dimensional subspace of  $\mathbb{R}^n$ , with  $m < n$ . Let  $P$  be the standard matrix for orthogonal projection into  $W$ . Is  $P$  invertible? Why or why not?

*$P$  is not invertible since  $\text{Null}(P)$  has dimension  $n-m > 0$ , and so is not one-to-one.*

12. A farmer with 1200 acres is considering planting three different crops, corn, soybeans, and oats. The farmer wants to use all 1200 acres. Corn seed costs \$20 per acre, soybean seed cost \$50 per acre, and oat seed costs \$12 per acre. The farmer has \$40,000 available to buy seed, and would like to use it all for this purpose. Set up a system of linear equations from this data whose solutions are the choices the farmer can make for quantities of each crop. Suppose that when the crops mature, the farmer brings in \$100 for each acre of corn, \$300 for each acre of soybeans, and \$80 for each acre of oats. How many of each crop should the farmer have bought to maximize profits?

$C = \text{acres of corn}$   $S = \text{acres of soybeans}$   $T = \text{acres of oats}$

$$C + S + T = 1200$$

$$20C + 50S + 12T = 40,000$$

$$\text{augmented mtr: } \left( \begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 20 & 50 & 12 & 40000 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 0 & 30 & -8 & 16000 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 38/30 & 20000/30 \\ 0 & 1 & -8/30 & 16000/30 \end{array} \right)$$

$$\Rightarrow \begin{aligned} 30C &= 20,000 - 38T \\ 30S &= 16,000 + 8T \end{aligned}$$

$$\begin{aligned} C \geq 0 &\Rightarrow 20,000 - 38T \geq 0 \\ &\Rightarrow T \leq 20,000/38 \end{aligned}$$

$$\text{earnings: } 100C + 300S + 80T$$

$$= \frac{100}{30} (20,000 - 38T) + \frac{300}{30} (16,000 + 8T) + 80T$$

$$= \frac{2,000,000}{30} + \frac{4,800,000}{30} + 42T \quad \text{maximal when } T \text{ maximal}$$

$$\Rightarrow T = 20,000/38 \text{ acres} \quad C = 0 \quad S = 16,000 + 8 \left( \frac{20,000}{38} \right) \text{ acres}$$

