Pre-Midterm Quiz
14.02.2012

No aids allowed. Time: 50 min.

Student Name:
(Please print) Last Given

Student #: Score:

PART A. MULTIPLE CHOICE QUESTIONS (2 marks for each question)[1]

A1. Evaluate $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin(2\theta)\, d\theta$.

(A) $\frac{\pi\sqrt{3}}{4}$ (B) 0 (C) $-\frac{\pi\sqrt{3}}{2}$ (D) $\frac{\pi}{6}$ (E) 1.

$$\int_{-\pi/6}^{\pi/6} \sin(2\theta)\, d\theta = -\frac{1}{2}\cos(2\theta)\Big]_{-\frac{\pi}{6}}^{\pi/6} = -\tfrac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\right) = 0.$$

Quick solution: If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$. $(a \in \mathbb{R})$

A2. The sequence $\{\ln(n+1) - \ln n\}_{n=1}^{\infty}$

(A) converges to 0 (B) converges to 1 (C) converges to e (D) converges to $\frac{1}{\sqrt{e}}$
(E) diverges.

$$\lim_{n\to\infty} (\ln(n+1) - \ln n) = \lim_{n\to\infty} \ln \frac{n+1}{n} = \ln(1) = 0.$$

[1]Use the last two pages for calculations.

A3. To evaluate $\displaystyle\int \frac{dx}{2x\sqrt{x^2+16}}$, the most appropriate substitution is

(A) $u = \tan(\frac{x}{4})$ (B) $u = \sin(\frac{x}{4})$ (C) $u = \tan^{-1}(\frac{x}{4})$ (D) $u = \sin^{-1}(\frac{x}{4})$ (E) $u = \sec(\frac{x}{4})$.

$$\sqrt{x^2+16} = \sqrt{x^2+4^2}$$

$$x = 4\tan(u) \to u = \tan^{-1}\left(\frac{x}{4}\right)$$

(see the table at page 478 of textbook).

A4. Which of the three series

$$\text{(i)} \sum_{n=1}^{\infty}\left(\frac{2n}{n^2} - \frac{3n}{n^3}\right) \qquad \text{(ii)} \sum_{n=1}^{\infty}\frac{n^2}{\sqrt[3]{n^{11}}} \qquad \text{(iii)} \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

converge?

(A) only (i) and (ii) converge (B) only (i) and (iii) converge
(C) only (ii) and (iii) converge (D) only (ii) converges (E) all three converge.

$$\text{(i)} \sum_{n=1}^{\infty}\left(\frac{2}{n} - \frac{3}{n^2}\right) = \underbrace{\sum_{n=1}^{\infty}\frac{2}{n}}_{\substack{\text{divergent} \\ \text{(p-series, } p=1)}} - \underbrace{\sum_{n=1}^{\infty}\frac{3}{n^2}}_{\substack{\text{convergent} \\ \text{(p-series, } p=2>1)}}$$

The sum of a convergent and a divergent series is divergent. (See exercise 11.2.83, page 713 of textbook.)

So it is divergent.

$$\text{(ii)} \sum_{n=1}^{\infty}\frac{n^2}{\sqrt[3]{n^{11}}} = \sum_{n=1}^{\infty}\frac{1}{n^{11/3-2}} \quad \left(\text{p-series, } p = \tfrac{11}{3}-2 > 1\right)$$

So it is convergent.

(iii) $\dfrac{1}{n(n+1)} \sim \dfrac{1}{n^2}$; Limit Comparison Test:

$$\lim_{n\to\infty} \frac{1}{n(n+1)} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = 1 \text{ (finite, nonzero)}$$

So: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p=2>1$) $\longrightarrow \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ converges.

A5. Which of the following statements most accurately describes the improper integral $\int_{-\infty}^{0} xe^{x^2}\, dx$?

(A) convergent (B) convergent to $\frac{1}{2}$ (C) convergent to e^{-4} (D) divergent
(E) divergent to $+\infty$.

$u = x^2 \to du = 2x\,dx$

$$\int_{-\infty}^{0} xe^{x^2}\,dx = \int_{+\infty}^{0} \frac{1}{2}e^u\,du = \frac{1}{2}e^u\Big]_{+\infty}^{0}$$

$$= \frac{1}{2}(e^0 - e^{+\infty}) = \frac{1}{2}(1-(+\infty)) = -\infty.$$

A6. The sequence $\left\{ \dfrac{(3n^2+1)\cdot 2^n}{n^{n/2}} \right\}_{n=1}^{\infty}$

(A) diverges (B) converges to 2 (C) converges to $\frac{2}{e}$ (D) converges to $\frac{1}{2}$
(E) converges to 0.

To be explained in problem-solving class.

A7. Which of the following statements most accurately describes the series $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$?

(A) divergent (B) conditionally convergent (C) absolutely convergent
(D) absolutely convergent and the sum is 1 (E) absolutely convergent and the sum is $\frac{1}{2}$.

Absolute Convergence: For n large enough, say $n \geq e$, we have $\ln(n) \geq 1$. So

$$\frac{\ln(n)}{n} \geq \frac{1}{n}.$$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, hence by comparison test $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is NOT absolutely convergent.

Alternating Series Test:

$f(n) = \frac{\ln(n)}{n}$ is finally decreasing, since

$$f'(x) = \left(\frac{\ln(x)}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2} \leq 0.$$

(If x is large enough, then $\ln(x) \geq 1$, then $1-\ln(x) \leq 0$.) So the sequence $\frac{\ln(n)}{n}$ becomes decreasing after some initial terms.

Also $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is convergent.

A8. Find the most appropriate partial fraction decomposition of $\dfrac{5x^4 - 3x^2 + 1}{(x^2-4)(x^2-x-6)(x^2+1)}$.

(A) $\frac{A}{x^2-4} + \frac{Bx+C}{x^2-x-6} + \frac{Dx+E}{x^2+1}$ (B) $\frac{A}{x-2} + \frac{B}{(x+2)^2} + \frac{C}{x-3} + \frac{Dx+E}{x^2+1}$

(C) $\frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2-x-6} + \frac{Ex+F}{x^2+1}$ (D) none of the above.

Decompose the denominator:

$$x^2-4 = (x-2)(x+2)$$

$$x^2-x-6 = (x+2)(x-3)$$

$$x^2+1 \rightsquigarrow \text{irreducible} \quad (\Delta = -4 < 0).$$

So we get

$$\frac{5x^4 - 3x^2 + 1}{(x-2)(x+2)^2(x-3)(x^2+1)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{x-3} + \frac{Ex+F}{x^2+1}.$$

So none of the forms is correct.

PART B. SHORT ANSWER QUESTIONS (show all your work)

B9. (5 marks) Evaluate $\int \frac{\ln x}{\sqrt{x}}\, dx$.

Integration by parts:

$u = \ln(x) \rightarrow du = \frac{1}{x}\, dx$

$dv = \frac{1}{\sqrt{x}}\, dx \rightarrow v = 2\sqrt{x}\, dx$

$$\int \frac{\ln(x)}{\sqrt{x}}\, dx = 2\sqrt{x}\ln(x) - \int \frac{2}{\sqrt{x}}\, dx$$

$$= 2\sqrt{x}\ln(x) - 4\sqrt{x} + C$$

$$= 2\sqrt{x}\,(\ln(x) - 2) + C.$$

B10. (5 marks) Evaluate $\displaystyle\int \frac{\cos x + \sin x}{\sin(2x)}\, dx$.

$$I = \int \frac{\cos x + \sin x}{2 \sin x \cos x}\, dx$$

$$= \frac{1}{2}\left(\int \frac{1}{\sin x}\, dx + \int \frac{1}{\cos x}\, dx \right)$$

$$= \frac{1}{2}\left(\int \operatorname{cosec} x\, dx + \int \sec x\, dx \right)$$

$$= \frac{1}{2}\left(\ln|\operatorname{cosec} x - \operatorname{cotan} x| + \ln|\sec x + \tan x| \right) + C$$

$$= \frac{1}{2} \ln\left| \frac{\operatorname{cosec} x - \operatorname{cotan} x}{\sec x + \tan x} \right| + C.$$

optional

B11. (6 marks) Determine whether the series $\sum_{n=1}^{\infty} \frac{(\sin n)^n + 2^{n/2}}{\sqrt[3]{n! + 4^n}}$ converges or diverges. Justify your answer.

Comparison Test:

$$\left| \frac{(\sin n)^n + 2^{n/2}}{\sqrt[3]{n! + 4^n}} \right| = \frac{|(\sin n)^n + 2^{n/2}|}{\sqrt[3]{n! + 4^n}} \leq \frac{|(\sin n)^n| + |2^{n/2}|}{\sqrt[3]{n! + 4^n}}$$

$$\leq \frac{1 + 2^{n/2}}{\sqrt[3]{n!}}$$

Now we test the convergence of

$$\sum_{n=1}^{\infty} \frac{1 + 2^{n/2}}{\sqrt[3]{n!}}$$ by the Ratio Test:

$$\lim_{n\to\infty} \frac{1 + 2^{\frac{n+1}{2}}}{\sqrt[3]{(n+1)!}} \Bigg/ \frac{1 + 2^{\frac{n}{2}}}{\sqrt[3]{n!}} = \lim_{n\to\infty} \frac{1 + 2^{\frac{n+1}{2}}}{1 + 2^{n/2}} \frac{\sqrt[3]{n!}}{\sqrt[3]{(n+1)!}}$$

dominant terms factored

$$= \lim_{n\to\infty} \frac{2^{\frac{n+1}{2}} \left(\frac{1}{2^{\frac{n+1}{2}}} + 1 \right)}{2^{\frac{n}{2}} \left(\frac{1}{2^{n/2}} + 1 \right)} \frac{\sqrt[3]{n!}}{\sqrt[3]{n+1}\,\sqrt[3]{n!}} = \lim_{n\to\infty} \frac{2^{1/2}}{\sqrt[3]{n+1}} = 0$$

< 1.

So $\sum_{n=1}^{\infty} \frac{1 + 2^{n/2}}{\sqrt[3]{n!}}$ converges, thus our original

series is absolutely convergent, and in particular convergent.