FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

Some Definitions

Differential Equation: an equation that contains the derivatives of one or more dependent variables with respect to one or more independent variables.

Ordinary Equation: A differential equation with ordinary (as opposed to partial) derivatives of one or more dependent variables with respect to a *single* independent variable.

Order of an Equation: The highest derivative in an equation.

Linearity: An equation is linear if (a) all the powers of y and its derivatives is 1 (b) There are no nonlinear functions of y present in the equation and (c) The coefficients of y and its derivatives are functions of x only.

Examples: Determine the order and linearity of the following equations. If nonlinear, explain why.

$$(a)\,(1-x)y'''-3x\cos xy''-(3y+4)y'=6x$$

$$(b)\,x^3\frac{d^4y}{dx^4}+x^4\left(\frac{dy}{dx}\right)^3=\ln(xy)$$

$$(c)\,5xy^{(7)}+4x^7y''-\frac{y'}{x^2}=0$$

$$(d)\,x\frac{dy}{dx}+\frac{4}{y}=6$$

Solution:

(a) Order = 3. Linearity: Nonlinear due to coefficient of y'.
(b) Order = 4. Linearity: Nonlinear due to $(y')^3$ and ln(xy) – a nonlinear function of y.
(c) Order = 7. Linearity: Linear.
(d) Order = 1. Linearity: Nonlinear as (4/y) is a nonlinear function of y.

Solutions to Differential Equations: A function y = f(x) is a solution to a given differential equation if the equation is satisfied if you plug in y and its derivatives.

Example: Show that $y_1 = C_1\cos 3t$ and $y_2 = C_2\sin 3t$ are both solutions to the equation: Is the linear combination of y_1 and y_2 also a solution of $y'' + 9y = 0$?

Solution:

$$y_1 = C_1 \cos 3t$$
$$y_1' = -3C_1 \sin 3t$$
$$y_1'' = -9C_1 \cos 3t$$

plug in to check if equation is satisfied:

$$LHS: -9C_1 \cos 3t + 9(C_1 \cos 3t) = 0 = RHS$$

$$y_2 = C_2 \sin 3t$$
$$y_2' = 3C_2 \cos 3t$$
$$y_2'' = -9C_2 \sin 3t$$

plug in to check if equation is satisfied:

$$LHS: -9C_2 \sin 3t + 9(C_2 \sin 3t) = 0 = RHS$$

Linear Combination:

$$y = C_1 y_1 + C_2 y_2$$
$$y = C_1 \cos 3t + C_2 \sin 3t$$
$$y' = -3C_1 \sin 3t + 3C_2 \cos 3t$$
$$y_1'' = -9C_1 \cos 3t - 9C_2 \sin 3t$$

plug in to check if equation is satisfied:

$$LHS: -9C_1 \cos 3t - 9C_2 \sin 3t + 9(C_1 \cos 3t + C_2 \sin 3t) = 0 = RHS$$

Le Cartier 3410 Peel, Suite 505, Montreal QC, H3A 1H3

SOLVING FIRST ORDER DIFFERENTIAL EQUATIONS

Separable Equations

Separable equations are often the easiest ones to solve. Given a differential equation you should always first check it is separable before applying other techniques. Solving separable equations is a simple 3 step process:

Step 1: Look to separate the variables: Get the x's on one side and the y's on the other do that you end up with:

$$f(y)dy = f(x)dx$$

Note – a) you can't have dx or dy in the denominator.
b) an equation of the form dy/dx + p(x)y = f(x) must have p(x) and f(x) = constant for it to be separable

Step 2: Integrate both sides of the equation:

$$\int f(y)dy = \int f(x)dx$$

Step 3: Use initial conditions to solve for the constant of integration.

Examples: Solve the following differential equations

$(a)\ y\dfrac{dy}{dx} = 2xe^{y^2},\ y(0) = 1$

$(b)\ \dfrac{dy}{dx} = e^{3x+4y}$ ← usually on finals

$(c)\ (4x+3)y' - y = 0$

First order differential equations: Integrating Factor

As we know 1st order linear differential equations take the form $C_1(x)\,y' + C_2(x)y = g(x)$. Like separable equations, solving first order equations is a mechanical process and can be solved by following the following 5 steps.

Step 1: Ensure that the coefficient of y' is 1. That is, divide the entire equation by $C_1(x)$ to obtain the following form:

If these are constants then the separable

$$y' + p(x)y = f(x)$$

Step 2: Compute the integrating factor:

$$\mu = e^{\int p(x)dx}$$

Step 3: Multiply the entire equation by μ. By doing so the left hand side of the equation will be the derivative of μy (product rule backwards):

$$\frac{d}{dx}(\mu y) = \mu f(x)$$

Step 4: Integrate both sides and solve for y:

$$\mu y = \int \mu f(x)dx$$

$$y = \frac{1}{\mu}\int \mu f(x)dx$$

Step 5: Use initial conditions to solve for the constant of integration.

Notes – sometimes computing the integral is hard: leave your answer in terms of the integral.

- you do not need to include a constant of integration when computing μ.
- all terms in the final answer that tend to 0 as the independent variable tends to ∞ (i.e. terms with negative exponents) are known as *transient terms*. The rest are called *steady state* terms

Examples: Solve the following differential equations

$(a)\ (x+1)y'+(x+2)y = 2xe^{-x},\ y(0)=5$

$(b)\ \dfrac{dy}{dx}+xy = 2$

$(c)\ ydx = (ye^{y} - 2x)dy$

Exact Equations

The exact equation is a very particular type of equation and usually very easily recognized by its form:

$$M(x, y)dx + N(x, y)dy = 0$$

There are 2 methods for solving this equation The first method involves the following steps:

Step 1: Make sure that the equation is in the form above and ensure that it is in fact exact, that is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step 2:

$$Let\ \frac{\partial f}{\partial x} = M \qquad\qquad Let\ \frac{\partial f}{\partial y} = N$$

$$So\ f_1 = \int M dx + g(y) \qquad\qquad So\ f_2 = \int N dy + h(x)$$

Step 3: Now the answer is simply a the sum: $f_1 + f_2 = C$. The only trick here is to be sure to include common terms between f_1 and f_2 only once and you are set.

Method 2: Sometimes one of the two integrations are difficult, in this case method 2 is more convenient to use. The first step of both methods is the same, things differ from the 2^{nd} step onward:

Step 2:

$$Let\frac{\partial f}{\partial x} = M$$
$$So\ f = \int M dx + g(y)$$

Step 3: Use the answer above to solve the following equation. This will allow you to solve for g(y)

$$Let\frac{\partial f}{\partial y} = N$$

Step 4: Replace g(y) in step 3 and write your solution in the form f(x, y) = C

Examples: Solve the following differential equations

$$(a)\,(x+y)^2 dx + (2xy + x^2 - 1)dy = 0$$
$$(b)\,(2xy - ye^{xy})dx + (x^2 + 4y^3 - xe^{xy})dy = 0$$

Optional: Non Exact Equations – Making them Exact.

Sometimes non exact equations can be made exact by first multiplying the entire equation by an integrating factor. The following formulas allow you to calculate the integrating factor, as long as it is calculated to be a function of a single variable, the equation can be made exact.

for this, if only x's, then good

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \qquad \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

memorize

In the equations above M_y and N_x are used to denote the partial derivatives with respect to the subscript.

Examples: Solve the following differential equation

$$(a)\,xydx + (2x^2 + 3y^2 - 20)dy = 0$$

SOLUTIONS BY SUBSTITUTION:

Many differential equations that initially appear difficult can be simplified considerable by making a simple substitution. There are many such substitutions but three are most common.

TYPE 1: Ax + By + C

Step 1: Let u = Ax + By + C, then u' = A +By', solve for y' and substitute back into the equation. It should simplify into something you know how to solve.

Step 2: Solve the equation and substitute back for u.

TYPE 2: HOMOGENEOUS

Homogeneous equations usually take the form of exact equations, so after checking whether or not they are exact we can see if they are homogeneous and follow the steps outlined below.

$$M(x, y)dx + N(x, y)dy = 0$$

Step 1: Determine whether or not the equation is homogeneous by substituting all x's with λx and all y's with λy. Factor out the highest power of λ in both M and N and as long as you are able to re-attain your original equation, the equation is homogeneous.

Step 2: Substitute y = ux (or x = uy) and dy = udx + xdu (or dx = udy +ydu) and simplify. Usually if N(x, y) seems harder, make the substitution in brackets. The equation should simplify into a separable equation, which you can now solve.

Step 3: Substitute back and simplify your answer.

TYPE 3: BERNOULLI

The Bernoulli equation is the simplest to recognize. The equation takes a form similar to the first order linear differential equation except it has a nonlinear term on the right hand side:

$$\frac{dy}{dx} + p(x)y = f(x)y^n$$

Bernoulli equations are generally long to solve so you should practice many of these. They always simplify into first order linear differential equations. The steps to solve these equations are as follows:

Step 1: Make sure the equation is in the correct form (as above)

Step 2: Let $u = y^{1-n}$, solve for y and y' (remember to use the chain rule – you must have a u' term when you solve for y')

Step 3: Substitute and simplify. The equation should reduce to a first order linear differential equation. Solve.

Step 4: Substitute back and simplify.

Examples: Determine which of the following equations is homogeneous and solve it

$(a)\ (2xy - x^3)dx - ydy = 0$

$(b)\ (y^2 + xy)dx + x^2dy = 0$

Examples: Solve the following differential equations

$(a)\ \dfrac{dy}{dx} = y(xy^3 - 1)$

$(b)\ \dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

LINEAR MODELS: 1st Order

The whole motivation behind learning to solve differential equations is that many real life situations can be modelled using differential equations. Three such models that often show up are outlined below. This is not an exhaustive list of models and if you have time, you should look into solving other models such as population, growth and decay etc.

Newton's Law of Cooling

Solving cooling/warming problems amounts to solving the following differential equation:

$$\frac{dT}{dt} = k(T - T_m)$$

where k is a constant of proportionality (that you usually have to solve for first), T_m is the temperature of the surrounding environment, sometimes you have to assume a value for T_m. The solution to the above equation always takes the following form:

$$T(t) = Ce^{-kt} + T_m$$

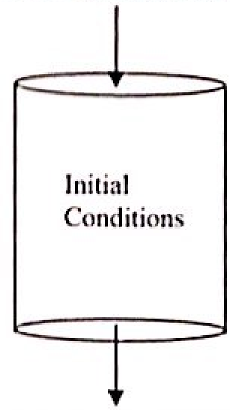
You can probably memorize this solution and plug in givens to solve of the constants but it is not difficult to derive. Solving such problems is demonstrated by means of the following example.

Example: A small metal bar, whose initial temperature is 20° is dropped into a container of boiling water. How long will it take the bar to reach 90° if it is known that its temperature increases 2° in one second?

Mixtures

In order to understand mixture problems, it is helpful to first understand the following diagram. If we can visualize and match the problem to the variables in the picture the problem becomes easier to solve. The differential equation related to the problem takes the form:

$$\frac{dy}{dt} = R_{in} - R_{out}$$



Once again the best way to understand how to solve this equation is by means of examples:

Examples:

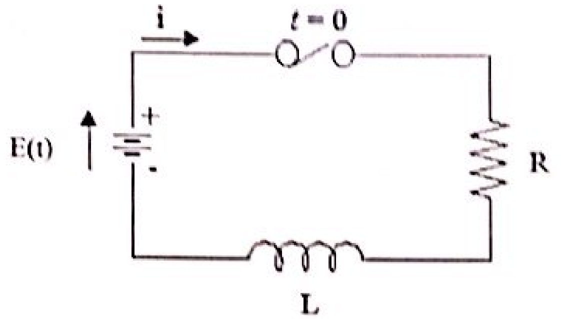
(a) The air in a meeting room of 15000ft^3 has a smoke content of 20 ppm. An air conditioner is turned on, which brings fresh air (with no smoke) into the room at a rate of 1200ft^3 per minute and forces the smoky air out at the same rate. How long will it take to reduce the smoke content to 5ppm?

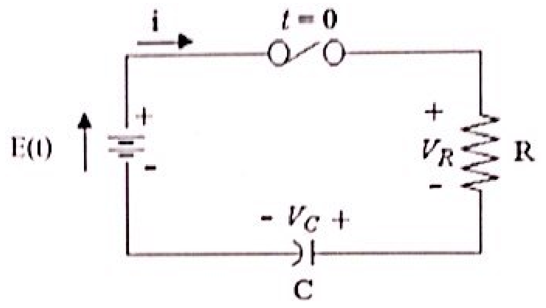
(b) A tank contains 200 gallons of brine that contains 3lbs of salt per gallon. Brine containing 2lbs of salt per gallon flows into the tank at a rate of 5 gallons per minute, while the mixture runs out at the same rate. Find the amount of salt as a function of time, what happens as $t \to \infty$?

(c) Five grams of chemical is dissolved in 100 litres of alcohol. A 10% mixture is added at the rate of 2 litres/min at the same time solution is being drained at the rate of 1 litre/min. Find an expression for the amount of chemical mixture at anytime. How much of the chemical is present after 30 min?

First Order Circuits

It turns out that circuits containing either a capacitor or an inductor (not both) can be modeled by a first order differential equation. Depending on the case the dependent variable is either the current in the circuit or the charge in the circuit. There are 2 specific series circuits that you will probably be responsible for, more complicated circuits are taught in ELEC 273/275. The circuits and the corresponding differential equations are given below.





$$L\frac{di}{dt} + Ri = E(t)$$

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Here, R, L and C represent the resistance, inductance and capacitance respectively. Basically they are numbers that you will plug into the correct equation. E(t) is the voltage as a function of time. Both of these equations are solved by either separation of variables (E(t) = 0) or by using an integrating factor.

Example: A 100-volt electromotive force is applied to an RC-series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge q(t) on the capacitor if q(0) = 0. Find i(t). [Hint: i = q']