PART A (17 marks)

1. [1 mark] Find the area of the region bounded by $y = \dfrac{1}{x}$ and the x-axis between $x = 2$ and $x = 5$.

A: $\ln 3$	B: $\ln 10$	C: $\dfrac{\ln 5}{\ln 2}$	D: $\ln\left(\dfrac{2}{5}\right)$	E: $\ln\left(\dfrac{5}{2}\right)$

Solution: The curve $y = \dfrac{1}{x}$ lies above the x-axis everywhere to the left of the y-axis, so we are asked to find the area of the region which lies below the curve $y = \frac{1}{x}$ and above the x-axis from $x = 2$ to $x = 5$. We get:

$$\text{Area } = \int_2^5 \frac{1}{x}\,dx = [\ln \mid x \mid]_2^5 = \ln 5 - \ln 2 = \ln\left(\frac{5}{2}\right)$$

2. [1 mark] Find the area of the region bounded by $y = e^x$ and $y = e^{-x}$ from $x = 0$ to $x = 1$.

A: $e + \dfrac{1}{e} - 2$	B: $e + \dfrac{1}{e} - 1$	C: $e + \dfrac{1}{e}$	D: $e - \dfrac{1}{e} + 1$	E: $e - \dfrac{1}{e} + 2$

Solution: Between $x = 0$ and $x = 1$ we have $e^x > 1$ and so $e^{-x} = \dfrac{1}{e^x} < 1$. Therefore the curve $y = e^x$ lies above the curve $y = e^{-x}$ everywhere between $x = 0$ and $x = 1$ (and in fact everywhere to the right of the y-axis). To find the area of the region which is bounded by $y = e^x$ on the top and $y = e^{-x}$ on the bottom, and by $x = 0$ on the left and $x = 1$ on the right, we use

$$\text{Area } = \int_{\text{left}}^{\text{right}} (\text{upper} - \text{lower})\,dx = \int_0^1 (e^x - e^{-x})\,dx$$

To find an antiderivative of e^{-x} we need the substitution rule, i.e. we use the fact that $\dfrac{e^{kx}}{k}$ is an antiderivative of e^{kx} for any constant k, and here the constant is $k = -1$. So we get $\frac{e^{-x}}{-1} = -e^{-x}$ as an antiderivative of e^{-x}. This gives

$$\begin{aligned} \text{Area } &= \int_0^1 (e^x - e^{-x})\,dx = \left[e^x - (-e^{-x})\right]_0^1 = \left[e^x + e^{-x}\right]_0^1 \\ &= (e^1 + e^{-1}) - (e^0 + e^{-0}) = e + \frac{1}{e} - (1+1) = e + \frac{1}{e} - 2 \end{aligned}$$

3. [1 mark] Find the area of the region bounded by $y = \sqrt{x}$, $x = 0$, $y = 1$ and $y = 2$.

A: $\dfrac{31}{5}$	B: $\dfrac{7}{3}$	C: $\dfrac{14}{3}$	D: $\dfrac{21}{2}$	E: 21

Solution: We have the region bounded by $y = \sqrt{x}$ on the right and the y-axis (i.e. the line $x = 0$) on the left, and by $y = 1$ on the bottom and $y = 2$ on the top. This area is most easily found using horizontal slicing, which means we need the curve forming the right boundary of the region to be expressed in $x = f(y)$ form. The curve $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ and so we get:

$$\begin{aligned} \text{Area } &= \int_{\text{bottom}}^{\text{top}} (\text{right} - \text{left})\,dy = \int_1^2 (y^2 - 0)\,dy = \int_1^2 y^2\,dy \\ &= \left[\frac{y^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \end{aligned}$$

4. [1 mark] Find the volume of the solid obtained when the region bounded by $y = x^2$, $y = 0$ and $x = 2$ is revolved about the x-axis.

A: $\frac{32}{5}\pi$	B: $\frac{64}{3}\pi$	C: $\frac{256}{3}\pi$	D: 6π	E: 8π

Solution: Since we are revolving the region about a horizontal axis we use vertical slicing. And since the line $y = 0$ is the x-axis, and so the axis of revolution is a boundary of the region, we use the Method of Disks. The region has the curve $y = x^2$ as its upper and left boundary, the x-axis as its lower boundary, and the line $x = 2$ as its right boundary. Therefore the region runs from $x = 0$ on the left (where $y = x^2$ intersects $y = 0$) to $x = 2$ on the right. A vertical slice of the region has length x^2 and so the radius of the disk is $r = x^2$. So with a slice of width dx giving a disk with height $h = dx$, the volume of the disk formed by revolving the slice is $\pi r^2 h = \pi (x^2)^2\,dx$. Therefore the volume of the entire solid formed by revolving the region is

$$\text{Volume } = \pi \int_0^2 (x^2)^2\,dx = \pi \int_0^2 x^4\,dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \pi \left(\frac{2^5}{5} - \frac{0^5}{5}\right) = \pi \left(\frac{32}{5} - 0\right) = \frac{32}{5}\pi$$

5. [1 mark] Determine an integral that represents the volume of the solid generated when the region bounded by $y = x^2$, $y = 0$ and $x = 2$ is revolved about the y-axis.

A: $\pi \int_0^4 y^4\,dy$	B: $\pi \int_0^4 (4 - y)\,dy$	C: $\pi \int_0^4 (2 - \sqrt{y})^2\,dy$
D: $\pi \int_0^2 (4 - y)\,dy$	E: $\pi \int_0^2 (2 - \sqrt{y})^2\,dy$	

Solution: We have the same region as in question 5, but now we are revolving the region about a vertical axis, so we need horizontal slicing. And since the axis of revolution (the y-axis, which is the line $x = 0$) is not a boundary of the region, we need the Method of Washers. Since the region lies entirely to the right of the y-axis, the left boundary of the region is part of the $x = \sqrt{y}$ half of the parabola $y = x^2$. And the right boundary is (as before) the line $x = 2$. So a horizontal slice of the region runs from $x = \sqrt{y}$ on the left to $x = 2$ on the right. Therefore the washer formed by revolving a small horizontal slice of the region about the y-axis has outer radius $R = 2$ and inner radius $r = \sqrt{y}$, and of course the height of the washer is $h = dy$. There are slices all the way up the region from $y = 0$ at the bottom to $y = 4$, the intersection of $x = 2$ with $y = x^2$, at the top, so to find the volume of the solid obtained by revolving the region about the y-axis we have

$$\text{Volume } = \pi \int_{\text{bottom}}^{\text{top}} (R^2 - r^2)\,dy = \pi \int_0^4 [(2)^2 - (\sqrt{y})^2]\,dy = \pi \int_0^4 (4 - y)\,dy$$

6. [1 mark] Which one of the following is equal to $\int x^2 \cos x\,dx$?

A: $x^2 \sin x - 2\int x \sin x\,dx$	B: $x^2 \cos x - 2\int x \cos x\,dx$	C: $\frac{x^3}{3} \sin x - \int x \sin x\,dx$
D: $x^2 \sin x - 2\int x \cos x\,dx$	E: $x \cos x - \int x \sin x\,dx$	

Solution: To find $\int x^2 \cos x\,dx$ we need Integration By Parts. Since $\cos x$ gets neither more nor less complicated whether differentiated or integrated, whereas x^2 gets more complicated when integrated and gets less complicated when differentiated, we choose x^2 as the term of the product in the integrand which will be differentiated. That is, we let $u = x^2$ and $dv = \cos x\,dx$, which gives $du = 2x\,dx$ and $v = \sin x$. Therefore we get

$$\int x^2 \cos x\,dx = \int u\,dv = uv - \int v\,du = x^2 \sin x - \int 2x \sin x\,dx = x^2 \sin x - 2\int x \sin x\,dx$$

7. [1 mark] Find $\int e^x \cos x\,dx$.

A: $e^x \sin x + C$	B: $e^x \sin x - e^x \cos x + C$	C: $e^x \sin x + e^x \cos x + C$
D: $\frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$	E: $\frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C$	

Solution: Again we need Integration By Parts. It doesn't matter which term in the product in the integrand is chosen as u and which is dv. We can let $u = e^x$ and $dv = \cos x\,dx$, which gives $du = e^x\,dx$ and $v = \sin x$. This gives

$$\int e^x \cos x\,dx = \int u\,dv = uv - \int v\,du = e^x \sin x - \int e^x \sin x\,dx$$

To find $\int e^x \sin x\,dx$ we need to use Integration By Parts again. Making choices similar to the first time we let $u = e^x$ and $dv = \sin x\,dx$, which gives $du = e^x\,dx$ and $v = -\cos x$. This gives

$$\int e^x \sin x\,dx = e^x(-\cos x) - \int (-\cos x)e^x dx = -e^x \cos x + \int e^x \cos x\,dx$$

Using this in what we had before we get

$$\begin{aligned}
\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\
\Rightarrow \int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\,dx\right) \\
\Rightarrow \int e^x \cos x\,dx &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx \\
\Rightarrow \int e^x \cos x\,dx + \int e^x \cos x\,dx &= e^x \sin x + e^x \cos x + C \\
\Rightarrow 2\int e^x \cos x\,dx &= e^x \sin x + e^x \cos x + C \\
\Rightarrow \int e^x \cos x\,dx &= \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C
\end{aligned}$$

(since of course one-half times an arbitrary constant is still just an arbitrary constant).

Notice that if we make the choices of u and dv the other way, so that in the first Integration By Parts we use $u = \cos x$ and $dv = e^x\,dx$, which gives $du = -\sin x\,dx$ and $v = e^x$ and then in the second Integration By Parts we use $u = \sin x$ and $dv = e^x\,dx$, which gives $du = \cos x\,dx$ and $v = e^x$, we get:

$$\begin{aligned}
\int e^x \cos x\,dx &= e^x \cos x - \int e^x(-\sin x)\,dx = e^x \cos x + \int e^x \sin x\,dx \\
&= e^x \cos x + e^x \sin x - \int e^x \cos x\,dx \\
\text{so that } 2\int e^x \cos x &= e^x \cos x + e^x \sin x + C \\
\text{and } \int e^x \cos x &= \frac{1}{2}\left(e^x \cos x + e^x \sin x\right) + C
\end{aligned}$$

So the same answer is obtained as in the first approach.

8. [1 mark] Evaluate $\int_1^2 xe^x\,dx$.

A: e^2+1	B: e^2-1	C: e^2	D: $3e^2+1$	E: $3e^2-1$

Solution: Using Integration By Parts to find an antiderivative of xe^x we let $u = x$ and $dv = e^x\,dx$ so that $du = dx$ and $v = e^x$ to get

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

so we see that $xe^x - e^x = e^x(x-1)$ is an antiderivative of xe^x. Therefore we get

$$\int_1^2 xe^x\,dx = [e^x(x-1)]_1^2 = e^2(2-1) - e^1(1-1) = e^2 - 0e = e^2$$

9. [1 mark] Evaluate $\int_1^2 x\ln x\,dx$.

A: $(2\ln 2) - \frac{5}{4}$	B: $(2\ln 2) - \frac{3}{4}$	C: $(4\ln 2) - 5$	D: $(4\ln 2) - 3$	E: None of A, B, C, D

Solution: Again we need Integration By Parts to find an antiderivative. We let $u = \ln x$ and $dv = x\,dx$, which gives $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$, so we get

$$\int x\ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)\,dx = \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2\ln x}{2} - \left(\frac{1}{2}\right)\left(\frac{x^2}{2}\right) + C$$

Therefore we see that $\frac{x^2\ln x}{2} - \left(\frac{1}{2}\right)\left(\frac{x^2}{2}\right) = \frac{x^2\ln x}{2} - \frac{x^2}{4}$ is an antiderivative of $x\ln x$ and so we get

$$\int_1^2 x\ln x\,dx = \left[\frac{x^2\ln x}{2} - \frac{x^2}{4}\right]_1^2 = \left(\frac{4\ln 2}{2} - \frac{4}{4}\right) - \left(\frac{1\ln 1}{2} - \frac{1}{4}\right) = (2\ln 2) - \frac{4}{4} - \frac{0}{2} + \frac{1}{4} = (2\ln 2) - \frac{3}{4}$$

10. [1 mark] Find the value of A if $\frac{2x+8}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$.

A: -2	B: 2	C: -4	D: 4	E: 3

Solution: Bringing the right hand side to a common denominator we get $\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$ and the numerator of this must be equal to the original numerator (since the denominator is the same as the original denominator), so we must have $A(x+1) + B(x-2) = 2x+8$. When $x = 2$ we get

$$A(2+1) + B(2-2) = 2(2) + 8 \quad \Rightarrow \quad 3A + 0B = 12 \quad \Rightarrow \quad A = 4$$

11. [1 mark] Find $\displaystyle\int_2^{\infty} \frac{1}{(2x+1)^2}\,dx$.

A: $\frac{1}{2}$	B: $\frac{1}{4}$	C: $\frac{1}{5}$	D: $\frac{1}{10}$	E: divergent

Solution:

$$\int_2^{\infty} \frac{1}{(2x+1)^2}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{1}{(2x+1)^2}\,dx$$

To find this integral we need the Substitution Rule. We let $u = 2x+1$, which gives $du = 2\,dx$ so that $dx = \left(\frac{1}{2}\right)\,du$. When $x = 2$ we have $u = 2(2)+1 = 5$, and when $x = b$ we have $u = 2b+1$, so we get

$$\begin{aligned}\int_2^{b} \frac{1}{(2x+1)^2}\,dx &= \int_5^{2b+1}\left(\frac{1}{u^2}\right)\left(\frac{1}{2}\right)du = \frac{1}{2}\int_5^{2b+1} u^{-2}\,du = \frac{1}{2}\left[\frac{u^{-1}}{-1}\right]_5^{2b+1} = \frac{1}{2}\left[-\frac{1}{u}\right]_5^{2b+1} \\ &= \frac{1}{2}\left[-\frac{1}{2b+1} - \left(-\frac{1}{5}\right)\right] = \frac{1}{2}\left[-\frac{1}{2b+1} + \frac{1}{5}\right] = \frac{1}{2}\left[\frac{1}{5} - \frac{1}{2b+1}\right] = \frac{1}{10} - \frac{1}{4b+2}\end{aligned}$$

As b approaches infinity, $4b+2$ also approaches infinity, so $\dfrac{1}{4b+2}$ approaches 0. Therefore we get

$$\int_2^{\infty} \frac{1}{(2x+1)^2}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{1}{(2x+1)^2}\,dx = \lim_{b\to\infty}\left[\frac{1}{10} - \frac{1}{4b+2}\right] = \frac{1}{10} - \lim_{b\to\infty}\left(\frac{1}{4b+2}\right) = \frac{1}{10} - 0 = \frac{1}{10}$$

12. [1 mark] Find $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$.

A: 3	B: $\frac{1}{3}$	C: -3	D: $-\frac{1}{3}$	E: divergent

Solution: We know that $\dfrac{e^{3x}}{3}$ is an antiderivative of e^{3x}. So we get

$$\begin{aligned}\int_{-\infty}^{0} e^{3x}\,dx &= \lim_{a\to-\infty}\int_a^0 e^{3x}\,dx = \lim_{a\to-\infty}\left[\frac{e^{3x}}{3}\right]_a^0 = \lim_{a\to-\infty}\left[\frac{e^{3(0)}}{3} - \frac{e^a}{3}\right] = \lim_{a\to-\infty}\left[\frac{1}{3} - \frac{e^a}{3}\right] \\ &= \frac{1}{3} - \lim_{a\to-\infty}\left[\frac{e^a}{3}\right] = \frac{1}{3} - \frac{\lim\limits_{a\to-\infty}(e^a)}{3} = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}\end{aligned}$$

13. [1 mark] Find $\displaystyle\int_1^{\infty} \frac{3}{x}\,dx$.

A: 0	B: $\frac{1}{3}$	C: 1	D: 3	E: divergent

Solution:

$$\int_1^{\infty} \frac{3}{x}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{3}{x}\,dx = \lim_{b\to\infty} 3\int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty}\left[3\ln|x|\right]_1^b = \lim_{b\to\infty} 3\left(\ln b - \ln 1\right) = 3\left[\lim_{b\to\infty}(\ln b)\right] - 0$$

As b approaches infinity, $\ln b$ also approaches infinity, so the integral diverges.

14. [1 mark] Find $\int_{-\infty}^{\infty} 2e^{3-|x|}dx$. (*Hint:* Consider what $|x|$ is, for $x < 0$ and for $x > 0$.)

A: e^3	B: $2e^3$	C: $4e^3$	D: $2e^{-3}$	E: divergent

Solution: We know that $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$ for any value c, provided both these integrals converge. Here, since we have $|x|$ appearing in the integrand function, and we know that $|x| = x$ for all $x \geq 0$, while $|x| = -x$ for all $x \leq 0$, it is useful to choose $c = 0$. So we use

$$\int_{-\infty}^{\infty} 2e^{3-|x|}dx = \int_{-\infty}^{0} 2e^{3-|x|}dx + \int_{0}^{\infty} 2e^{3-|x|}dx$$

We need to consider each of these integrals separately, and we re-express the integrand, replacing $|x|$ by $-x$ in the integral from $-\infty$ to 0, and replacing $|x|$ by x in the integral from 0 to ∞. (We will also use the fact that as $c \to -\infty$, $e^c \to 0$.) And we will need the substitution rule for both, letting u equal the exponent in each case. For the first integral we have $3 - |x| = 3 - (-x) = 3 + x$ so the substitution is $u = 3 + x$ which gives $du = dx$ and so we get $e^u = e^{3+x}$ as an antiderivative of e^{3+x}. This gives

$$\begin{aligned}\int_{-\infty}^{0} 2e^{3-|x|}dx &= 2\int_{-\infty}^{0} e^{3-(-x)}dx = 2\int_{-\infty}^{0} e^{3+x}dx = 2\left[\lim_{a\to-\infty}\int_{a}^{0} e^{3+x}dx\right] \\ &= 2\left[\lim_{a\to-\infty}\left(e^{3+x}\Big]_a^0\right)\right] = 2\left[\lim_{a\to-\infty}\left(e^{3+0} - e^{3+a}\right)\right] = 2e^3 - 2\left[\lim_{a\to-\infty} e^{3+a}\right]\end{aligned}$$

As a approaches $-\infty$, $3 + a$ also approaches $-\infty$, and so e^{3+a} approaches 0. Therefore we have

$$\int_{-\infty}^{0} 2e^{3-|x|}dx = 2e^3 - 2\left[\lim_{a\to-\infty} e^{3+a}\right] = 2e^3 - 2(0) = 2e^3$$

For the second integral we get $3 - |x| = 3 - x$ and so the substitution is $u = 3 - x$, which gives $du = -dx$ and so we get $-e^u = -e^{3-x}$ as an antiderivative of e^{3-x}. This gives

$$\begin{aligned}\int_{0}^{\infty} 2e^{3-|x|}dx &= 2\int_{0}^{\infty} e^{3-x}dx = 2\left[\lim_{b\to\infty}\int_{0}^{b} e^{3-x}dx\right] = 2\left[\lim_{b\to\infty}\left(-e^{3-x}\Big]_0^b\right)\right] \\ &= 2\left[\lim_{b\to\infty}\left(-e^{3-b} + e^{3-0}\right)\right] = 2e^3 - 2\left[\lim_{b\to\infty} e^{3-b}\right]\end{aligned}$$

As b approaches ∞, $3 - b$ approaches $-\infty$, and so e^{3-b} approaches 0. Therefore we have

$$\int_{0}^{\infty} 2e^{3-|x|}dx = 2e^3 - 2\left[\lim_{b\to\infty} e^{3-b}\right] = 2e^3 - 2(0) = 2e^3$$

Since both of the improper integrals which have one infinite limit of integration converge, then the improper integral from $-\infty$ to ∞ also converges. We get

$$\int_{-\infty}^{\infty} 2e^{3-|x|}dx = \int_{-\infty}^{0} 2e^{3-|x|}dx + \int_{0}^{\infty} 2e^{3-|x|}dx = 2e^3 + 2e^3 = 4e^3$$

15. [1 mark] If $f(x, y, z) = z^x e^{\cos y}$, find $f(2, \pi, -1)$.

A: e	B: $-e$	C: $\frac{1}{e}$	D: $-\frac{1}{e}$	E: 1

Solution:

$$f(2, \pi, -1) = (-1)^2 e^{\cos\pi} = (1)e^{-1} = \frac{1}{e}$$

16. [1 mark] Find $f_x(2,3)$ where $f(x,y) = x^3y - 3y^2 + 2x - y$.

A: -11	B: 14	C: 56	D: 46	E: 38

Solution: When we find the partial with respect to x, we treat y as a constant, so any term that doesn't have an x in it has derivative 0.

$$f_x(x,y) = \frac{\partial}{\partial x}(x^3y - 3y^2 + 2x - y) = y(3x^2) - 0 + 2 - 0 = 3x^2y + 2$$

Therefore we have $f_x(2,3) = 3(2^2)(3) + 2 = 9(4) + 2 = 36 + 2 = 38$.

17. [1 mark] If $f(x,y,z) = xyz\ln(xyz)$, find $f_y(x,y,z)$.

A: 1	B: $\ln(xyz) + \dfrac{1}{xyz}$	C: $xz\ln(xyz) + \dfrac{1}{y}$	D: $xz\ln(xyz) + xz$	E: $xz + \dfrac{1}{xz}$

Solution: This time, when differentiating with respect to y, we treat both x and z as constants. Since we have a product of 2 terms which both involve y, we need the product rule.

$$\begin{aligned} f_y(x,y,z) &= \left[\frac{\partial}{\partial y}(xyz)\right][\ln(xyz)] + (xyz)\left[\frac{\partial}{\partial y}[\ln(xyz)]\right] \\ &= (xz)(1)[\ln(xyz)] + (xyz)\frac{\frac{\partial}{\partial y}(xyz)}{xyz} = xz\ln(xyz) + xz \end{aligned}$$

PART B (8 marks)

18. [2 marks] Set up a *single* integral that represents the area of the region bounded by $x = y^2$ and $x = 2y + 3$.

Solution: The curves $x = y^2$ and $x = 2y + 3$ intersect when $y^2 = 2y + 3$, i.e. when $y^2 - 2y - 3 = 0$. Since $y^2 - 2y - 3 = (y+1)(y-3)$ we see that the curves intersect at $y = -1$ and at $y = 3$. To find the area of the region bounded by $x = y^2$ and $x = 2y + 3$ we use horizontal slicing. Within the interval $[-1, 3]$ the function $x = y^2$ has a smaller value than the function $x = 2y + 3$ (for instance, at $y = 0$ we have $0^2 = 0$ and $0 + 3 = 3$), so $x = 2y + 3$ is further to the right than $x = y^2$. Therefore we get

$$\text{Area } = \int_{\text{bottom}}^{\text{top}} (\text{rightmost} - \text{leftmost})\,dy = \int_{-1}^{3} [(2y+3) - y^2]\,dy = \int_{-1}^{3} (2y + 3 - y^2)\,dy$$

19. [2 marks] Set up an integral that represents the volume of the solid obtained when the region bounded by $y = \ln x$, $y = 1$ and $x = 5$ is revolved about the x-axis.

Solution: Since the region is being revolved about the x-axis, a horizontal axis, we use vertical slicing and integrate with respect to x. Since the x-axis (the line $y = 0$) is not a boundary of the region, we use the Method of Washers. The line $y = 1$ intersects the curve $y = \ln x$ at $x = e$, and of course we know that $5 > e$. So the region lies between $y = \ln x$ and $y = 2$, from $x = e$ to $x = 5$. For $x > e$ we have $\ln x > 1$ and so $y = \ln x$ is the upper curve between $x = e$ and $x = 5$. Therefore a vertical slice of the region has upper edge $y = \ln x$ and lower edge $y = 1$ and thus when the slice is revolved about the x-axis it forms a washer with outer radius $R = \ln x$ and inner radius $r = 1$. We get the volume of the solid generated by revolving the region about the x-axis as

$$\text{Volume } = \pi\int_{\text{left}}^{\text{right}} (R^2 - r^2)\,dx = \pi\int_{e}^{5} [(\ln x)^2 - (1)^2]\,dx = \pi\int_{e}^{5} [(\ln x)^2 - 1]\,dx$$

20. [2 marks] Find $\int \frac{4}{x^2+2x-3}\,dx$.

Solution: The integrand is a rational function with at degree 0 polynomial in the numerator and a degree 2 polynomial in the denominator, so we can use partial fractions. We start by factoring the denominator polynomial.

$$\frac{4}{x^2+2x-3} = \frac{4}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1} = \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}$$

We see that we need $A(x-1)+B(x+3)=4$. When $x=-3$ we get $A(-3-1)+B(-3+3)=4$, so $-4A=4$ and therefore $A=-1$. And when $x=1$ we get $A(1-1)+B(1+3)=4$ so $4B=4$ and therefore $B=1$. Now we can evaluate the integral, using the partial fraction decomposition:

$$\int \frac{4}{x^2+2x-3}\,dx = \int \left(\frac{-1}{x+3}+\frac{1}{x-1}\right)dx = -\int \frac{1}{x+3}\,dx + \int \frac{1}{x-1}\,dx = -\ln|x+3| + \ln|x-1| + C = \ln\left|\frac{x-1}{x+3}\right| + C$$

21. [2 marks] Consider the function $f(x,y)=e^{x/y}$.

(a) Find f_y.

Solution: We need the Chain Rule:

$$f_y = \frac{\partial}{\partial y}(e^{x/y}) = e^{x/y}\left[\frac{\partial}{\partial y}\left(\frac{x}{y}\right)\right] = e^{x/y}(x)\left[\frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right] = xe^{x/y}\left[\frac{\partial}{\partial y}(y^{-1})\right] = xe^{x/y}(-y^{-2}) = -\frac{xe^{x/y}}{y^2}$$

(b) Find $\frac{\partial}{\partial x}(f_y)$. **Do NOT simplify your answer.**

Solution: We have $f_y = -\frac{xe^{x/y}}{y^2} = \left(-\frac{x}{y^2}\right)e^{x/y}$, so we can use the product rule to find the partial of this function with respect to x. We get

$$\begin{aligned}
\frac{\partial}{\partial x}(f_y) &= \frac{\partial}{\partial x}\left[\left(-\frac{x}{y^2}\right)e^{x/y}\right] = \left[\frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right)\right](e^{x/y}) + \left(-\frac{x}{y^2}\right)\left[\frac{\partial}{\partial x}(e^{x/y})\right] \\
&= \left(-\frac{1}{y^2}\right)\left[\frac{\partial}{\partial x}(x)\right](e^{x/y}) - \frac{x}{y^2}\left\{e^{x/y}\left[\frac{\partial}{\partial x}\left(\frac{x}{y}\right)\right]\right\} \\
&= -\frac{e^{x/y}}{y^2} - \frac{x}{y^2}(e^{x/y})\left(\frac{1}{y}\right)\left[\frac{\partial}{\partial x}(x)\right] = -\frac{e^{x/y}}{y^2} - \frac{xe^{x/y}}{y^3} = -\frac{e^{x/y}}{y^2}\left(1+\frac{x}{y}\right)
\end{aligned}$$

(Any form of this in which the differentiating was completed should have received full marks.)