## PART A (18 marks)

1. [1 mark] Simplify $\frac{1}{5}\left(125\,x^9\right)^{1/3}$.

| A: $25x^3$ | B: $5x^2$ | C: $\frac{1}{5}x^6$ | D: $x^3$ | E: $5x^3$ |
|---|---|---|---|---|

*Solution:*

$$\frac{1}{5}\left(125\,x^9\right)^{1/3} = \frac{1}{5}\left[(5^3)^{1/3}(x^9)^{1/3}\right] = \frac{1}{5}\left[5^{3/3}x^{9/3}\right] = \frac{5x^3}{5} = x^3$$

2. [1 mark] Express $27^{2/3} = 9$ in logarithmic form.

| A: $\log_9 27 = \frac{2}{3}$ | B: $\log_{27} 9 = \frac{2}{3}$ | C: $\log_{27} 9 = \frac{3}{2}$ | D: $\log_{27}\frac{2}{3} = 9$ | E: $\log_{\frac{2}{3}} 27 = 9$ |
|---|---|---|---|---|

*Solution:* We know that $y = b^x$ says the same thing as $x = \log_b y$, so with $b = 27$, $x = \frac{2}{3}$ and $y = 9$ we see that $9 = 27^{2/3}$ says $\frac{2}{3} = \log_{27} 9$.

3. [1 mark] Solve for $x$ if $\log_2 x + \log_2(x-2) = 3$.

| A: $x = 4$ | B: $x = 2$ | C: $x = -2$ | D: $x = 4$ and $x = -2$ | E: $x = 4$ and $x = 2$ |
|---|---|---|---|---|

*Solution:*

$$\log_2 x + \log_2(x-2) = 3 \quad \Rightarrow \quad \log_2[x(x-2)] = 3 \quad \Rightarrow \quad \log_2(x^2-2x) = 3$$

And of course, this says the same thing as $x^2 - 2x = 2^3$ so we have $x^2 - 2x = 8$. This is satisfied when $x^2 - 2x - 8 = 0$, i.e. when $(x-4)(x+2) = 0$. We see that $x^2 - 2x = 8$ both when $x = 4$ and when $x = -2$. However, we know that $\log_2 x$ is only defined when $x > 0$, so only $x = 4$ satisfies $\log_2 x + \log_2(x-2) = 3$.

4. [1 mark] If $f(x) = xe^{(x^2)}$, find $f'(2)$.

| A: $4e^2$ | B: $5e^4$ | C: $4e^4$ | D: $8e^2$ | E: $9e^4$ |
|---|---|---|---|---|

*Solution:* We need the product rule, with the chain rule.

$$f'(x) = (1)e^{(x^2)} + x\left[e^{(x^2)}\left(\frac{d}{dx}(x^2)\right)\right] = e^{(x^2)} + xe^{(x^2)}(2x) = e^{(x^2)}(1+2x^2)$$

Therefore we see that $f'(2) = e^{(2^2)}[1 + 2(2^2)] = e^4[1+2(4)] = 9e^4$.

5. [1 mark] If $f(x) = \log_5 x$, find $f'(x)$.

| A: $\frac{1}{x\ln 5}$ | B: $\frac{x}{\ln 5}$ | C: $\frac{1}{5\ln x}$ | D: $\frac{1}{5x}$ | E: $\frac{\ln 5}{x}$ |
|---|---|---|---|---|

*Solution:* We know that $\frac{d}{dx}(\log_b x) = \frac{1}{x\ln b}$ so for $b = 5$ we get $f'(x) = \frac{d}{dx}(\log_5 x) = \frac{1}{x\ln 5}$.

6. [1 mark] Find the slope of the tangent line to the graph of $y = 3^{(x^2)}$ at the point $(1, 3)$.

| | | | | |
|---|---|---|---|---|
| A: $3 \ln 3$ | B: $6$ | C: $6 \ln 3$ | D: $3$ | E: $3 \ln 6$ |

*Solution:* We have the graph of $y = f(x)$ for $f(x) = 3^{(x^2)}$. The slope of the tangent line is given by

$$f'(x) = \frac{d}{dx}\left(3^{(x^2)}\right) = \left[3^{(x^2)} \ln 3\right]\left[\frac{d}{dx}(x^2)\right] = \left[3^{(x^2)} \ln 3\right](2x)$$

When $x = 1$ we have $f'(1) = \left[3^{(1^2)} \ln 3\right][2(1)] = 2(3^1 \ln 3) = 6 \ln 3$ and so the slope of the tangent line at the point $(1, 3)$ is $6 \ln 3$.

7. [1 mark] Evaluate $\cos\left(-\frac{4\pi}{3}\right)$.

| | | | | |
|---|---|---|---|---|
| A: $\frac{\sqrt{3}}{2}$ | B: $-\frac{1}{2}$ | C: $-\frac{\sqrt{3}}{2}$ | D: $\sqrt{3}$ | E: $\frac{1}{2}$ |

*Solution:* We know that $\cos(-x) = \cos x$, and the angle $\frac{4\pi}{3}$ radians is in the third quadrant, in which the value of cosine is negative. The reference angle is $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$. Therefore we get

$$\cos\left(-\frac{4\pi}{3}\right) = \cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

(Notice that we would come to the same conclusion using the fact that $-\frac{4\pi}{3}$ is in the second quadrant, where cosine is also negative, since the reference angle would still be $\frac{\pi}{3}$.)

8. [1 mark] If $f(x) = e^{3x} \sin 2x$, find $f'(x)$.

| | | |
|---|---|---|
| A: $6e^{3x} \sin 2x$ | B: $e^{3x}(3 \sin 2x + 2 \cos 2x)$ | C: $6e^x \sin x \cos x$ |
| D: $e^{3x}(3 \sin 2x - 2 \cos 2x)$ | E: $6e^x(\cos 2x + \sin 2x)$ | |

*Solution:* We need the product rule, with the chain rule in both terms of the sum.

$$f'(x) = \left[e^{3x}\left(\frac{d}{dx}(3x)\right)\right](\sin 2x) + e^{3x}\left[(\cos 2x)\left(\frac{d}{dx}(2x)\right)\right] = 3e^{3x} \sin 2x + e^{3x}(2 \cos 2x) = e^{3x}(3 \sin 2x + 2 \cos 2x)$$

9. [1 mark] If $f(x) = \cos^3 x$, find $f'\left(\frac{\pi}{6}\right)$.

| | | | | |
|---|---|---|---|---|
| A: $\frac{9}{4}$ | B: $1$ | C: $-\frac{3\sqrt{3}}{8}$ | D: $-\frac{3\sqrt{3}}{4}$ | E: $-\frac{9}{8}$ |

*Solution:*

$$f'(x) = \frac{d}{dx}(\cos^3 x) = \frac{d}{dx}[(\cos x)^3] = 3(\cos x)^2\left[\frac{d}{dx}(\cos x)\right] = (3 \cos^2 x)(-\sin x) = -3 \sin x \cos^2 x$$

And we know that $\sin\frac{\pi}{6} = \frac{1}{2}$, while $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so we get

$$f'\left(\frac{\pi}{6}\right) = -3\left(\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{6}\right)^2 = -3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 = -3\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = -\frac{9}{8}$$

10. [1 mark] Find $\int (x^2 - e^x + \sec^2 x)\, dx$.

| | |
|---|---|
| A: $\frac{x^3}{3} - e^x + \frac{\sec^3 x}{3} + C$ | B: $2x - e^x + 2\sec^2 x \tan x + C$ |
| C: $\frac{x^3}{3} - e^x + \tan x + C$ | D: $\frac{x^3}{3} - \ln\lvert x\rvert + \tan x + C$ |
| E: $\frac{x^3}{3} - e^x + \frac{\sec^3 x}{3\sec x \tan x} + C$ | |

*Solution:* We know that $\frac{x^3}{3}$ is an antiderivative of $x^2$, and that $\tan x$ is an antiderivative of $\sec^2 x$, so we get

$$\int (x^2 - e^x + \sec^2 x)\, dx = \int x^2\, dx - \int e^x\, dx + \int \sec^2 x\, dx = \frac{x^3}{3} - e^x + \tan x + C$$

11. [1 mark] Find $\int x\left(x^4 + \frac{1}{x^2}\right)\, dx$.

| | | |
|---|---|---|
| A: $\frac{x^6}{6} + \ln\lvert x\rvert + C$ | B: $x^5 + \frac{1}{x} + C$ | C: $x^6 + e^x + C$ |
| D: $\frac{\frac{x^7}{7} + x}{\frac{x^2}{2}} + C$ | E: $\frac{x^2}{2}\left(\frac{x^5}{5} - \frac{1}{x}\right) + C$ | |

*Solution:*

$$\int x\left(x^4 + \frac{1}{x^2}\right)\, dx = \int \left(x^5 + \frac{1}{x}\right)\, dx = \frac{x^6}{6} + \ln\lvert x\rvert + C$$

12. [1 mark] If $f'(x) = 5x^4 - 12x^5$ and $f(1) = 4$, find $f(-1)$.

| | | | | |
|---|---|---|---|---|
| A: 1 | B: 2 | C: 3 | D: 4 | E: 5 |

*Solution:*

$$f(x) = \int f'(x)\, dx = \int (5x^4 - 12x^5)\, dx = x^5 - 2x^6 + C$$

This gives $f(1) = 1^5 - 2(1^6) + C = 1 - 2 + C = C - 1$ and so we must have $C - 1 = 4$, which gives $C = 5$. Thus $f(x) = x^5 - 2x^6 + 5$, so $f(-1) = (-1)^5 - 2(-1)^6 + 5 = -1 - 2 + 5 = 2$.

13. [1 mark] Find $\int e^x \cos(e^x)\, dx$.

| | | |
|---|---|---|
| A: $e^x \sin(e^x) + C$ | B: $\cos(e^x) + C$ | C: $\sin(e^x)\cos(e^x) + C$ |
| D: $\sin(e^x) + C$ | E: $e^x \cos(e^x) + C$ | |

*Solution:* Since the angle in the trig function is more complicated than just $x$, we need the substitution rule. Letting $u = e^x$ (i.e. using "let $u$ = the angle function") we have $du = e^x\, dx$, so we get

$$\int e^x \cos(e^x)\, dx = \int [\cos(e^x))]\, e^x\, dx = \int \cos u\, du = \sin u + C = \sin(e^x) + C$$

14. [1 mark] Find $\int \frac{1}{x\sqrt{\ln x}}\,dx$.

| | | | | |
|---|---|---|---|---|
| A: $\frac{1}{\sqrt{\ln x}}+C$ | B: $x\sqrt{\ln x}+C$ | C: $\frac{\sqrt{\ln x}}{x}+C$ | D: $2\sqrt{\ln x}+C$ | E: $\frac{2}{3}(\ln x)^{3/2}+C$ |

*Solution:* We have an integrand function which is not something we recognize as a derivative, so we suspect that we will need the substitution rule. Since we see something being done to $\ln x$, we try $u=\ln x$. This gives $du=\left(\frac{1}{x}\right)dx$, and we do see $\frac{1}{x}$ in the integrand function, so this looks promising. Replacing both $\ln x$ and $\frac{1}{x}\,dx$, we have

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{\ln x}}\left(\frac{1}{x}\right)dx = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2}+C = 2\sqrt{u}+C = 2\sqrt{\ln x}+C$$

15. [1 mark] Evaluate $\int_1^2 \frac{x+1}{x}\,dx$.

| | | | | |
|---|---|---|---|---|
| A: $1+\ln 2$ | B: $-1$ | C: $\ln 2$ | D: $1$ | E: $2$ |

*Solution:*

$$\int_1^2 \frac{x+1}{x}\,dx = \int_1^2 \left(\frac{x}{x}+\frac{1}{x}\right)dx = \int_1^2 \left(1+\frac{1}{x}\right)dx = [x+\ln|x|]_1^2 = (2+\ln 2)-(1+\ln 1) = (2-1)+\ln 2-0 = 1+\ln 2$$

16. [1 mark] Evaluate $\int_{-2}^0 f(x)\,dx$ if we are given that $\int_0^3 f(x)\,dx=3$ and $\int_{-2}^3 f(x)\,dx=5$.

| | | | | |
|---|---|---|---|---|
| A: $-2$ | B: $8$ | C: $-8$ | D: $4$ | E: $2$ |

*Solution:* We know that $\int_a^b f(x)\,dx+\int_b^c f(x)\,dx=\int_a^c f(x)\,dx$, so we see that

$$\int_{-2}^3 f(x)\,dx = \int_{-2}^0 f(x)\,dx+\int_0^3 f(x)\,dx \quad\Rightarrow\quad 5=\int_{-2}^0 f(x)\,dx+3 \quad\Rightarrow\quad \int_{-2}^0 f(x)\,dx = 5-3=2$$

17. [1 mark] Evaluate $\int_0^{\frac{\pi}{4}} \sin x\cos x\,dx$.

| | | | | |
|---|---|---|---|---|
| A: $\frac{1}{2}$ | B: $\frac{1}{4}$ | C: $\frac{1}{8}$ | D: $\frac{1}{16}$ | E: $1$ |

*Solution:* If we let $u=\sin x$ then we get $du=\cos x\,dx$ and we have $u=\sin 0=0$ when $x=0$ whereas $u=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ when $x=\frac{\pi}{4}$. This gives

$$\int_0^{\pi/4} \sin x\cos x\,dx = \int_0^{1/\sqrt{2}} u\,du = \left[\frac{u^2}{2}\right]_0^{1/\sqrt{2}} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{2}-\frac{0^2}{2} = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

(*Notice:* Using the substitution $v=\cos x$ instead works out to the same answer.)

18. [1 mark] Find the average value of $f(x) = ex^{e-1} + \dfrac{1}{e}$ over the interval $0 \le x \le e$.

| A: $e^e - e$ | B: $\dfrac{e^{e-1}+1}{e}$ | C: $\dfrac{e^e+1}{e}$ | D: $e^e$ | E: $e^e + 1$ |
|---|---|---|---|---|

*Solution:* We know that the average value of the function $f(x)$ on the interval $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

In this case, we have $f(x) = ex^{e-1} + \dfrac{1}{e}$ and the interval $[0, e]$. Realizing that of course $e$ is a constant, we have

$$\begin{aligned} f_{ave} &= \frac{1}{e-0}\int_0^e \left(ex^{e-1} + \frac{1}{e}\right) dx = \frac{1}{e}\left[(e)\left(\frac{x^{(e-1)+1}}{(e-1)+1}\right) + \left(\frac{1}{e}\right)(x)\right]_0^e = \frac{1}{e}\left[\frac{ex^e}{e} + \frac{x}{e}\right]_0^e \\ &= \left[\frac{x^e}{e} + \frac{x}{e^2}\right]_0^e = \left(\frac{e^e}{e} + \frac{e}{e^2}\right) - \left(\frac{0^e}{e} + \frac{0}{e^2}\right) = \frac{e^e}{e} + \frac{1}{e} - (0+0) = \frac{e^e+1}{e} \end{aligned}$$

## PART B (7 marks)

19. [2 marks] Find $f'(2)$ if $f(x) = \ln(x^2+1)$.

*Solution:* We need the chain rule for $f'(x)$:

$$f'(x) = \frac{1}{x^2+1}\left[\frac{d}{dx}(x^2+1)\right] = \frac{\frac{d}{dx}(x^2+1)}{x^2+1} = \frac{2x}{x^2+1}$$

Therefore we get $f'(2) = \dfrac{2(2)}{2^2+1} = \dfrac{4}{5}$.

20. [2 marks] If $f(x) = x^{x+1}$, find $f'(1)$.

*Solution:* For $f(x) = x^{x+1}$ we have the form $(f(x))^{g(x)}$ and so we must use logarithmic differentiation. Letting $y = f(x)$ we have

$$\begin{aligned} y = x^{x+1} \quad \Rightarrow \quad \ln y &= \ln x^{x+1} = (x+1)(\ln x) \\ \Rightarrow \quad \frac{y'}{y} &= (1)(\ln x) + (x+1)\left(\frac{1}{x}\right) = \ln x + \frac{x+1}{x} \\ \Rightarrow \quad f'(x) = y' &= \left(\ln x + \frac{x+1}{x}\right) y = \left(\ln x + \frac{x+1}{x}\right)(x^{x+1}) \\ \Rightarrow \quad f'(1) &= \left(\ln 1 + \frac{1+1}{1}\right)(1^{1+1}) = (0+2)(1) = 2 \end{aligned}$$

21. [3 marks] Evaluate $\displaystyle\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx$.

*Solution:* We need the substitution rule. Letting $u = x^3 + 1$ (i.e. using "let $u$ equal the thing under the square root") we have $du = 3x^2\,dx$ and so $x^2\,dx = \left(\frac{1}{3}\right)du$. When $x = 0$ we have $u = 0^3 + 1 = 1$ and when $x = 2$ we get $u = 2^3 + 1 = 9$ and so we have

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \int_1^9 \frac{1}{\sqrt{u}}\left(\frac{1}{3}\right)du = \frac{1}{3}\int_1^9 u^{-1/2}\,du = \frac{1}{3}\left[\frac{u^{1/2}}{1/2}\right]_1^9 = \left[\frac{1}{3}\times 2\sqrt{u}\right]_1^9 = \frac{2\sqrt{9}-2\sqrt{1}}{3} = \frac{6-2}{3} = \frac{4}{3}$$