PART A (35 marks)

A1. If $f(x) = \dfrac{e^x}{\ln x}$, find $f'(x)$.

A: 0	B: xe^x	C: $\dfrac{e^x}{\ln x}$	D: $\dfrac{e^x - \frac{1}{x}}{(\ln x)^2}$	E: $\dfrac{e^x\left((\ln x) - \frac{1}{x}\right)}{(\ln x)^2}$

Solution: We use the quotient rule:

$$f'(x) = \frac{\left[\frac{d}{dx}(e^x)\right](\ln x) - e^x\left[\frac{d}{dx}(\ln x)\right]}{(\ln x)^2} = \frac{e^x(\ln x) - e^x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{e^x\left((\ln x) - \frac{1}{x}\right)}{(\ln x)^2}$$

A2. Find $f'(x)$ where $f(x) = \log_2 x^3$.

A: $\dfrac{3}{x}$	B: $\dfrac{1}{x^3}$	C: $\dfrac{1}{x^3 \ln 2}$	D: $\dfrac{3\ln 2}{x}$	E: $\dfrac{3}{x \ln 2}$

Solution:

$$f'(x) = \frac{d}{dx}(\log_2 x^3) = \frac{d}{dx}(3\log_2 x) = 3\left[\frac{d}{dx}(\log_2 x)\right] = 3\left(\frac{1}{x\ln 2}\right) = \frac{3}{x\ln 2}$$

A3. Find the slope of the tangent line to the graph of $y = x^2 + 2^x$ at the point $(0, 1)$ on the graph.

A: 1	B: 3	C: $\ln 2$	D: $2 + \ln 2$	E: $\dfrac{1}{\ln 2}$

Solution: For $f(x) = x^2 + 2^x$ we have $f'(x) = 2x + 2^x \ln 2$ and so the slope of the tangent line at the point with $x = 0$ and $y = 1$ is

$$f'(0) = 2(0) + 2^0 \ln 2 = 0 + 1\ln 2 = \ln 2$$

A4. If $f'(x) = \dfrac{1}{x} + e^x$ and $f(1) = 0$, what is $f(x)$?

A: 0	B: $\ln\lvert x\rvert + e^x + 1$	C: $\ln\lvert x\rvert + e^x - e$	D: $\dfrac{1}{x} + e^x - 1 - e$	E: $-\dfrac{1}{x^2} + e^x + 1 - e$

Solution:

$$f(x) = \int\left(\frac{1}{x} + e^x\right)dx = \ln\lvert x\rvert + e^x + C$$

We see that $f(1) = \ln 1 + e^1 + C = 0 + e + C = e + C$ and so $f(1) = 0$ gives $C = -e$. Therefore $f(x) = \ln\lvert x\rvert + e^x - e$.

A5. Evaluate $\displaystyle\int_0^{\sqrt{\pi/2}} x\sin(x^2)\,dx$.

A: 0	B: $\dfrac{1}{2}$	C: 1	D: $\dfrac{\pi}{2}$	E: $\cos\sqrt{\dfrac{\pi}{2}}$

Solution: We need the substitution rule. Letting $t = x^2$ we get $dt = 2x\,dx$ so that $x\,dx = \left(\frac{1}{2}\right)dt$. When $x = 0$ we have $t = 0^2 = 0$ and when $x = \sqrt{\frac{\pi}{2}}$ we have $t = \left(\sqrt{\frac{\pi}{2}}\right)^2 = \frac{\pi}{2}$ so we get

$$\int_0^{\sqrt{\pi/2}} x\sin(x^2)\,dx = \frac{1}{2}\int_0^{\pi/2}\sin t\,dt = \frac{1}{2}[-\cos t]_0^{\pi/2} = \left(\frac{1}{2}\right)\left[-\cos\left(\frac{\pi}{2}\right) + \cos 0\right] = \left(\frac{1}{2}\right)(-0 + 1) = \frac{1}{2}$$

A6. Evaluate $\displaystyle\int_e^1 \frac{\sqrt{\ln x}}{x}\,dx$.

A: $-\frac{2}{3}$	B: $\frac{2}{3}$	C: $1-\sqrt{e}$	D: $1-e$	E: $1-\frac{1}{e^2}$

Solution: We need a substitution. Let $t=\ln x$ so that $dt=\frac{1}{x}\,dx$. When $x=e$ we have $t=\ln e=1$ and when $x=1$ we have $t=\ln 1=0$. Therefore we have

$$\int_e^1 \frac{\sqrt{\ln x}}{x}\,dx=\int_1^0 \sqrt{t}\,dt=\int_1^0 t^{1/2}dt=\left[\frac{t^{3/2}}{3/2}\right]_1^0=\left[\frac{2\sqrt{t^3}}{3}\right]_1^0=\frac{2\sqrt{0^3}}{3}-\frac{2\sqrt{1^3}}{3}=-\frac{2}{3}$$

A7. Find $\displaystyle\int x\sin x\,dx$.

A: $\sin x+C$	B: $\cos x+C$	C: $-\cos(x^2)+C$	D: $\sin x+\cos x+C$	E: $-x\cos x+\sin x+C$

Solution: We need integration by parts. Letting $u=x$ and $dv=\sin x\,dx$ we have $du=dx$ and $v=-\cos x$ so we get

$$\int x\sin x\,dx=\int u\,dv=uv-\int v\,du=x(-\cos x)-\int(-\cos x)\,dx=-x\cos x+\int\cos x\,dx=-x\cos x+\sin x+C$$

A8. Evaluate $\displaystyle\int_0^{\ln 2} xe^x\,dx$.

A: $\ln 2+1$	B: $2\ln 2-1$	C: $\ln 2-1$	D: $2\ln 2+3$	E: 1

Solution: We need integration by parts to find an antiderivative. Letting $u=x$ and $dv=e^x\,dx$ we get $du=dx$ and $v=e^x$ so we see that

$$\int xe^x\,dx=xe^x-\int e^x\,dx=xe^x-e^x+C=e^x(x-1)+C$$

That is, $e^x(x-1)$ is an antiderivative of xe^x, so we get

$$\int_0^{\ln 2} xe^x\,dx=[e^x(x-1)]_0^{\ln 2}=e^{\ln 2}(\ln 2-1)-e^0(0-1)=2(\ln 2-1)-1(-1)=2\ln 2-2+1=2\ln 2-1$$

A9. If $\dfrac{2x+3}{x^2-1}=\dfrac{A}{x+1}+\dfrac{B}{x-1}$, find the value of A.

A: -1	B: $-\frac{1}{2}$	C: $\frac{5}{2}$	D: $\frac{1}{2}$	E: 1

Solution:

$$\frac{2x+3}{x^2-1}=\frac{2x+3}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}=\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

so we need $A(x-1)+B(x+1)=2x+3$. When $x=-1$ we get

$$A(-1-1)+B(-1+1)=2(-1)+3\Rightarrow -2A+0B=-2+3\Rightarrow -2A=1\Rightarrow A=-\frac{1}{2}$$

A10. Find the average value of the function $f(x) = 3x^2$ on the interval $[1, 3]$.

A: 12	B: 13	C: 24	D: 26	E: 0

Solution:

$$f_{ave} = \frac{1}{3-1}\int_1^3 3x^2\,dx = \left[\frac{x^3}{2}\right]_1^3 = \frac{3^3}{2} - \frac{1^3}{2} = \frac{27}{2} - \frac{1}{2} = \frac{26}{2} = 13$$

A11. Find the area of the region bounded by the curves $y = \ln x$, $y = x$, $x = 1$ and $x = 2$.

A: $\frac{5}{2} - 2\ln 2$	B: $\frac{9}{2} - 2\ln 2$	C: $2\ln 2 - 1$	D: $2\ln 2 - 3$	E: $1 - \ln 2$

Solution: The curve $y = \ln x$ crosses the x-axis at $x = 1$ and lies below $y = x$ everywhere. That is, at $x = 1$ we have $\ln x = 0$ so $\ln x < x$. And at $x = 2$ we have $\ln x = \ln 2$ which is less than 2, since $2 < e^2$ and we know that $\ln e^2 = 2$. So $y = x$ is the upper "curve" and $y = \ln x$ is the lower curve everywhere in the interval from $x = 1$ to $x = 2$. Therefore we see that

$$\text{Area } = \int_1^2 (x - \ln x)\,dx = \left[\frac{x^2}{2}\right]_1^2 - \int_1^2 \ln x\,dx$$

We need integration by parts to find an antiderivative of $\ln x$. Letting $u = \ln x$ and $dv = dx$ we have $du = \frac{1}{x}\,dx$ and $v = x$ so we get

$$\int \ln x\,dx = (\ln x)x - \int x\left(\frac{1}{x}\right)\,dx = x\ln x - \int dx = x\ln x - x + C$$

Therefore we see that $x\ln x - x$ is an antiderivative of $\ln x$, so we get

$$\begin{aligned}\text{Area} &= \left[\frac{x^2}{2}\right]_1^2 - \int_1^2 \ln x\,dx = \left[\frac{x^2}{2} - (x\ln x - x)\right]_1^2 = \left(\frac{2^2}{2} - (2\ln 2 - 2)\right) - \left(\frac{1^2}{2} - (1\ln 1 - 1)\right) \\ &= 2 - 2\ln 2 + 2 - \frac{1}{2} + 1(0) - 1 = 4 - 2\ln 2 - \frac{1}{2} - 1 = 3 - \frac{1}{2} - 2\ln 2 = \frac{5}{2} - 2\ln 2\end{aligned}$$

A12. Which one of the following is an integral that represents the area of the region bounded by the curves $x = y^2$ and $x = 2 - y^2$?

A: $\int_{-1}^{1}(2 - 2y^2)\,dy$	B: $\int_{-1}^{1}(2y^2 - 2)\,dy$	C: $\int_{0}^{1}(2 - 2y^2)\,dy$	D: $\int_{0}^{1}(2y^2 - 2)\,dy$	E: $\int_{0}^{1}(2 - y^2)\,dy$

Solution: We need to find the y-values at which the curves intersect. We see that

$$y^2 = 2 - y^2 \Rightarrow 2y^2 = 2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

Between $y = -1$ and $y = 1$, we can use $y = 0$ to see that $x = 2 - y^2 = 2$ is further to the right than $x = y^2 = 0$. Therefore the area of the region is given by

$$\text{Area } = \int_{\text{bottom}}^{\text{top}} (\text{rightmost} - \text{leftmost})\,dy = \int_{-1}^{1} [(2 - y^2) - y^2]\,dy = \int_{-1}^{1} (2 - 2y^2)\,dy$$

A13. Which one of the following is an integral that represents the volume of the solid generated when the region bounded by $x = \cos y$ and $x = 0$, between $y = 0$ and $y = \dfrac{\pi}{2}$, is revolved about the y-axis?

A: $\pi \int_0^1 \cos^2 y \, dy$	B: $\pi \int_0^{\pi/2} \cos^2 y \, dy$	C: $\pi \int_0^1 \sin^2 x \, dx$	D: $\pi \int_0^{\pi/2} \sin^2 x \, dx$	E: None of A, B, C or D

Solution: The curve $x = \cos y$ crosses the x-axis (i.e. the line $y = 0$) at $x = \cos 0 = 1$. The first place above the x-axis that this curve crosses the y-axis is at $y = \frac{\pi}{2}$. That is, we know that $\cos \frac{\pi}{2} = 0$, and that $\cos y > 0$ for all y-values inside $\left(0, \frac{\pi}{2}\right)$. Therefore the curve $x = \cos y$ lies to the right of the line $x = 0$ (i.e. the y-axis) everywhere in the region.

When a slice of the region is revolved about the y-axis, since this axis *is* a boundary of the region, a disk is formed. The disk has radius equal to the length of the slice, which is the distance from $x = 0$ to $x = \cos y$, so $r = \cos y$. Therefore the volume of the solid obtained by revolving the whole region about the y-axis is given by

$$\text{Volume } = \pi \int_{\text{bottom}}^{\text{top}} r^2 \, dy = \pi \int_0^{\pi/2} (\cos y)^2 \, dy = \pi \int_0^{\pi/2} \cos^2 y \, dy$$

A14. Find the volume of the solid generated when the region bounded by $y = x$, $x = 0$ and $y = 2$ is revolved about the x-axis.

A: $\dfrac{\pi}{3}$	B: $\dfrac{4\pi}{3}$	C: $\dfrac{8\pi}{3}$	D: $\dfrac{11\pi}{3}$	E: $\dfrac{16\pi}{3}$

Solution: The region has the line $y = 2$ as upper boundary and the line $x = 0$ as left boundary, with the line $y = x$ forming the lower and right boundaries. Since the x-axis is *not* a boundary of the region, when a vertical slice of the region is revolved about this axis it forms a washer. The outer radius of the washer is given by the upper edge of the slice which is $y = 2$, so $R = 2$. The inner radius of the washer is given by the lower edge of the slice which is $y = x$, so $r = x$. The region extends from $x = 0$ to $x = 2$ (where $y = x$ intersects $y = 2$). Therefore the volume of the solid generated by revolving the entire region is

$$\begin{aligned}\text{Volume} &= \pi \int_{\text{left}}^{\text{right}} (R^2 - r^2) \, dx = \pi \int_0^2 (2^2 - x^2) \, dx = \pi \int_0^2 (4 - x^2) \, dx = \pi \left[4x - \frac{x^3}{3}\right]_0^2 \\ &= \pi \left[\left(4(2) - \frac{2^3}{3}\right) - \left(4(0) - \frac{0^3}{3}\right)\right] = \pi \left(8 - \frac{8}{3}\right) - 0 = \frac{16\pi}{3}\end{aligned}$$

A15. Evaluate the improper integral $\displaystyle\int_0^\infty e^{-x} \, dx$, if it converges.

A: -1	B: 0	C: 1	D: e	E: The integral diverges.

Solution: We see that (or use a substitution to find that) $-e^{-x}$ is an antiderivative of e^{-x}. Therefore

$$\int_0^\infty e^{-x} \, dx = \lim_{b \to \infty} \int_0^b e^{-x} \, dx = \lim_{b \to \infty} \left[-e^{-x}\right]_0^b = \lim_{b \to \infty} (-e^{-b} + e^0) = 1 - \lim_{b \to \infty} \left(\frac{1}{e^b}\right)$$

As $b \to \infty$, $e^b \to \infty$ as well, so $\dfrac{1}{e^b} \to 0$. Thus we get

$$\int_0^\infty e^{-x} \, dx = 1 - \lim_{b \to \infty} \left(\frac{1}{e^b}\right) = 1 - 0 = 1$$

A16. If $f(x, y, z) = \ln(x^3 + y^2 + z)$, find $f_x(1, 1, 1)$.

A: $\ln 3$	B: $\frac{1}{3}$	C: $\frac{2}{3}$	D: 1	E: 2

Solution:

$$f_x(x, y, z) = \frac{\partial}{\partial x}[\ln(x^3 + y^2 + z)] = \frac{\frac{\partial}{\partial x}(x^3 + y^2 + z)}{x^3 + y^2 + z} = \frac{3x^2 + 0 + 0}{x^3 + y^2 + z} = \frac{3x^2}{x^3 + y^2 + z}$$

Therefore $f_x(1, 1, 1) = \dfrac{3(1)^2}{1^3 + 1^2 + 1} = \dfrac{3}{1 + 1 + 1} = \dfrac{3}{3} = 1$.

A17. Find $f_y(2, 2)$ if $f(x, y) = 2xy^2 + x$.

A: 16	B: 17	C: 9	D: 24	E: 25

Solution:

$$f_y(x, y) = \frac{\partial}{\partial y}[2xy^2 + x] = 2x\left[\frac{\partial}{\partial y}(y^2)\right] + 0 = 2x(2y) = 4xy$$

Therefore $f_y(2, 2) = 4(2)(2) = 16$.

A18. Find $f_{xy}(x, y)$ where $f(x, y) = xe^y + x^2y$.

A: $xe^y + x^2$	B: $e^y + 2xy$	C: $xe^y + e^y + x^2 + xy$	D: $xe^y + 2xy$	E: $e^y + 2x$

Solution: We differentiate first with respect to x and then with respect to y.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}[xe^y + x^2y] = e^y\left[\frac{\partial}{\partial x}(x)\right] + y\left[\frac{\partial}{\partial x}(x^2)\right] = e^y(1) + y(2x) = e^y + 2xy \\ \text{so } f_{xy}(x, y) &= \frac{\partial}{\partial y}[f_x(x, y)] = \frac{\partial}{\partial y}[e^y + 2xy] = e^y + 2x(1) = e^y + 2x \end{aligned}$$

Use the following information to answer questions A19, A20 and A21.

$$f(x, y) = xy - x^2y - xy^2$$
$$f_x(x, y) = y - 2xy - y^2, \quad f_y(x, y) = x - x^2 - 2xy$$
$$f_{xx}(x, y) = -2y, \quad f_{xy}(x, y) = 1 - 2x - 2y, \quad f_{yy}(x, y) = -2x$$

A19. Which one of the following is true for the point $(1, 0)$?

A: $(1, 0)$ is not a critical point of $f(x, y)$.	B: $f(x, y)$ has a local maximum at $(1, 0)$.
C: $f(x, y)$ has a local minimum at $(1, 0)$.	D: $f(x, y)$ has a saddle point at $(1, 0)$.
E: The second partials test yields no information.	

Solution: $f_x(1, 0) = 0 - 2(1)(0) - 0^2 = 0$ and $f_y(1, 0) = 1 - 1^2 - 2(1)(0) = 1 - 1 = 0$ so we see that $(1, 0)$ *is* a critical point of f. The function for the second partials test is

$$D(x, y) = [f_{xx}(x, y)][f_{yy}(x, y)] - [f_{xy}(x, y)]^2 = (-2y)(-2x) - (1 - 2x - 2y)^2 = 4xy - (1 - 2x - 2y)^2$$

At the point $(1, 0)$ we get

$$D(1, 0) = 4(1)(0) - [1 - 2(1) - 2(0)]^2 = 0 - (-1)^2 = -1$$

Since $D(1, 0) < 0$, $f(x, y)$ has a saddle point at $(1, 0)$.

A20. Which one of the following is true for the point $(1, 1)$?

A: $(1, 1)$ is not a critical point of $f(x, y)$.	B: $f(x, y)$ has a local maximum at $(1, 1)$.
C: $f(x, y)$ has a local minimum at $(1, 1)$.	D: $f(x, y)$ has a saddle point at $(1, 1)$.
E: The second partials test yields no information.	

Solution: $f_x(1, 1) = 1 - 2(1)(1) - 1^2 = -2 \neq 0$ so $(1, 1)$ is not a critical point of $f(x, y)$.

A21. Which one of the following is true for the point $\left(\frac{1}{3}, \frac{1}{3}\right)$?

A: $\left(\frac{1}{3}, \frac{1}{3}\right)$ is not a critical point of $f(x, y)$.	B: $f(x, y)$ has a local maximum at $\left(\frac{1}{3}, \frac{1}{3}\right)$.
C: $f(x, y)$ has a local minimum at $\left(\frac{1}{3}, \frac{1}{3}\right)$.	D: $f(x, y)$ has a saddle point at $\left(\frac{1}{3}, \frac{1}{3}\right)$.
E: The second partials test yields no information.	

Solution: $f_x\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} - 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{2}{9} - \frac{1}{9} = 0$ and similarly $f_y\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} - \frac{1}{9} - \frac{2}{9} = 0$ so we see that $\left(\frac{1}{3}, \frac{1}{3}\right)$ *is* a critical point of f. For the second partials test, using what we found for $D(x, y)$ above we have

$$D\left(\frac{1}{3}, \frac{1}{3}\right) = 4\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left[1 - \frac{2}{3} - \frac{2}{3}\right]^2 = \frac{4}{9} - \left(-\frac{1}{3}\right)^2 = \frac{4}{9} - \frac{1}{9} = \frac{3}{9}$$

Since $D\left(\frac{1}{3}, \frac{1}{3}\right) > 0$, while $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0$ we see that $f(x, y)$ has a local maximum at $\left(\frac{1}{3}, \frac{1}{3}\right)$.

A22. Find all critical points of $f(x, y) = \frac{x^4}{4} - xy + \frac{y^4}{4}$.

A: $(0, 0)$ only	B: $(1, 1)$ only	C: $(-1, 1)$ only
D: $(0, 0)$ and $(1, 1)$only	E: $(0, 0), (1, 1)$ and $(-1, -1)$	

Solution:

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}\left[\frac{x^4}{4} - xy + \frac{y^4}{4}\right] = \frac{4x^3}{4} - y(1) + 0 = x^3 - y \\ f_y(x, y) &= \frac{\partial}{\partial y}\left[\frac{x^4}{4} - xy + \frac{y^4}{4}\right] = 0 - x(1) + \frac{4y^3}{4} = y^3 - x \end{aligned}$$

For (x, y) to be a critical point of f it must be true that $f_x(x, y) = 0$ and also $f_y(x, y) = 0$. For $f_x(x, y) = 0$ we need $x^3 - y = 0$ so $y = x^3$. For $f_y(x, y) = 0$ we need $y^3 - x = 0$ so $x = y^3$. In order to have both true at the same time we must have

$$x = y^3 = (x^3)^3 \Rightarrow x = x^9 \Rightarrow x = 0 \text{ or } x^8 = 1 \Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1$$

Since we must have $y = x^3$ we see that when $x = 0$, $y = 0$ and when $x = 1$, $y = 1$ and when $x = -1$, $y = -1$. Therefore the critical points of f are $(0, 0), (1, 1)$ and $(-1, -1)$.

A23. Find a local minimum value of the function $f(x, y) = x^3 - 3x + y^2 - 8y + 11$.

A: -7	B: -3	C: -1	D: 1	E: -18

Solution: We use the second partials test. First, we need to find the critical points of f, i.e. all points at which both first partials are 0.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(x^3 - 3x + y^2 - 8y + 11) = 3x^2 - 3 \\ f_y(x, y) &= \frac{\partial}{\partial y}(x^3 - 3x + y^2 - 8y + 11) = 2y - 8 \end{aligned}$$

For $f_x(x, y) = 0$ we need

$$3x^2 - 3 = 0 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

and for $f_y(x, y) = 0$ we need

$$2y - 8 = 0 \Rightarrow 2y = 8 \Rightarrow y = 4$$

Therefore both first partials are 0 only when $x = \pm 1$ and $y = 4$ so the only critical points are $(1, 4)$ and $(-1, 4)$.

We need to find the D function for the second partials test, so we need the second partials of f.

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x}(f_x(x,y)) = \frac{\partial}{\partial x}(3x^2-3) = 6x \\ f_{yy}(x,y) &= \frac{\partial}{\partial y}(f_y(x,y)) = \frac{\partial}{\partial y}(2y-8) = 2 \\ f_{xy}(x,y) &= \frac{\partial}{\partial y}(f_x(x,y)) = \frac{\partial}{\partial y}(3x^2-3) = 0 \end{aligned}$$

This gives the function for the second partials test as

$$\begin{aligned} D(x,y) &= [f_{xx}(x,y)][f_{yy}(x,y)] - [f_{xy}(x,y)]^2 = (6x)(2) - (0)^2 = 12x \\ \text{so } D(1,4) &= 12(1) = 12 \\ \text{and } D(-1,4) &= 12(-1) = -12 \end{aligned}$$

For the point $(x, y) = (1, 4)$ we see that $D(1, 4) > 0$ and we have $f_{xx}(1, 4) = 6(1) = 6 > 0$ and so there is a local minimum of f at $(1, 4)$. For the point $(x, y) = (-1, 4)$ we have $D(-1, 4) < 0$, so there is a saddle point at $(-1, 4)$. Therefore the *only* local minimum of f is at $(1, 4)$. Therefore the only local minimum value of f is

$$f(1, 4) = 1^3 - 3(1) + 4^2 - 8(4) + 11 = 1 - 3 + 16 - 32 + 11 = -7$$

A24. To maximize $f(x, y, z) = 2x - y + z$ subject to the constraint that $xy + xz + yz = 3$ using Lagrange's method, which system of equations must be solved?

A: $2 + \lambda y + \lambda z = 0$; $-1 + \lambda x + \lambda z = 0$; $1 + \lambda x + \lambda y = 0$; $xy + xz + yz - 3 = 0$	B: $2 - y + \lambda z = 0$; $1 + \lambda x + z = 0$; $2x + \lambda y + 1 = 0$; $xy + xz - 3 = 0$	C: $\lambda y + \lambda z = 0$; $\lambda x + \lambda z = 0$; $\lambda x + \lambda y = 0$; $xy + xz + yz - 3 = 0$
D: $2 + y + z = 0$; $-1 + x + z = 0$; $1 + x + y = 0$; $xy + xz + yz - 3 = 0$	E: None of A, B, C or D.	

Solution: The constraint has the form $g(x, y, z) = 3$ with $g(x, y, x) = xy + xz + yz$. We form the Lagrange function, $F(x, y, z, \lambda)$ by adding λ times $g(x, y, z) - 3$ to the function $f(x, y, z)$:

$$F(x, y, z, \lambda) = 2x - y + z + \lambda(xy + xz + yz - 3)$$

The system of equations which must be solved is the system which requires all first partials of this Lagrange function to be 0. The first partials of F are

$$\begin{aligned} F_x(x,y,z,\lambda) &= 2 + \lambda(y+z) = 2 + \lambda y + \lambda z \\ F_y(x,y,z,\lambda) &= -1 + \lambda(x+z) = -1 + \lambda x + \lambda z \\ F_z(x,y,z,\lambda) &= 1 + \lambda(x+y) = 1 + \lambda x + \lambda y \\ F_\lambda(x,y,z,\lambda) &= 1(xy+xz+yz-3) = xy + xz + yz - 3 \end{aligned}$$

Therefore the system which needs to be solved is

$$2 + \lambda y + \lambda z = 0; -1 + \lambda x + \lambda z = 0; 1 + \lambda x + \lambda y = 0; \text{ and } xy + xz + yz - 3 = 0$$

A25. Consider the problem of finding the maximum and minimum values of the function $3x - 2y + 5$ subject to the constraint that $3x^2 + 2y^2 = 80$ using the method of Lagrange multipliers. After solving the appropriate system of equations, which points are identified as candidates for the locations of the maximum and minimum function values?

A: $(4,4)$ and $(-4,4)$	B: $(4,-4)$ and $(-4,-4)$	C: $(4,4)$ and $(-4,-4)$
D: $(4,-4)$ and $(-4,4)$	E: $(4,4),(4,-4),(-4,4)$ and $(-4,-4)$	

Solution: The lagrange function is $F(x, y, \lambda) = 3x - 2y + 5 + \lambda(3x^2 + 2y^2 - 80)$ which has first partials

$$\begin{aligned} F_x(x,y,\lambda) &= 3 + \lambda(6x) \\ F_y(x,y,\lambda) &= -2 + \lambda(4y) \\ F_\lambda(x,y,\lambda) &= 3x^2 + 2y^2 - 80 \end{aligned}$$

This gives the system to be solved as

$$3 + 6x\lambda = 0 \text{ and } -2 + 4y\lambda = 0 \text{ and } 3x^2 + 2y^2 - 80 = 0$$

For $F_x(x,y,\lambda) = 0$ we need $6x\lambda = -3$ so $x \neq 0$ and we must have $\lambda = -\dfrac{1}{2x}$. Similarly, for $F_y(x,y,\lambda) = 0$ we need $4y\lambda = 2$, so $y \neq 0$ and we must have $\lambda = \dfrac{1}{2y}$. Therefore having both these first partials be 0 at the same time requires that

$$-\frac{1}{2x} = \frac{1}{2y} \Rightarrow -2y = 2x \Rightarrow y = -x$$

And for $F_\lambda(x,y,\lambda) = 0$ we need $3x^2 + 2y^2 = 80$, so for this to be true at the same time that both of the other first partials are 0, i.e. when $y = -x$, we must have

$$3x^2 + 2(-x)^2 = 80 \Rightarrow 3x^2 + 2x^2 = 80 \Rightarrow 5x^2 = 80 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$$

Since we must have $y = -x$ we see that when $x = 4$ we need $y = -4$ and when $x = -4$ we need $y = 4$. Therefore $(4,-4)$ and $(-4,4)$ are the only candidate points.

A26. Find the maximum value of the function $f(x,y) = 2x - y$ subject to the constraint $x^2 + y^2 = 20$.

A: $4\sqrt{5}$	B: 6	C: 8	D: 10	E: 20

Solution: We use Lagrange's method. The Lagrange function is $F(x,y,\lambda) = 2x - y + \lambda(x^2 + y^2 - 20)$ which has first partials

$$F_x(x,y,\lambda) = 2 + \lambda(2x) \text{ and } F_y(x,y,\lambda) = -1 + \lambda(2y) \text{ and } F_\lambda(x,y,\lambda) = x^2 + y^2 - 20$$

For $F_x(x,y,\lambda) = 0$ we need

$$2 + 2x\lambda = 0 \Rightarrow 2x\lambda = -2 \Rightarrow \lambda = -\frac{1}{x}$$

and for $F_y(x,y,\lambda) = 0$ we need

$$-1 + 2y\lambda = 0 \Rightarrow 2y\lambda = 1 \Rightarrow \lambda = \frac{1}{2y}$$

so for both of these to be 0 at the same time we must have

$$-\frac{1}{x} = \frac{1}{2y} \Rightarrow x = -2y$$

To have $F_\lambda(x,y,0) = 0$ as well, we must have $x^2 + y^2 = 20$ when $x = -2y$, which gives

$$(-2y)^2 + y^2 = 20 \Rightarrow 4y^2 + y^2 = 20 \Rightarrow 5y^2 = 20 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Requiring $y = \pm 2$ with $x = -2y$ gives 2 candidate points: $(-4,2)$ and $(4,-2)$. We need to find the values of $f(x,y)$ at these two points:

$$f(-4,2) = 2(-4) - 2 = -10 \text{ and } f(4,-2) = 2(4) - (-2) = 10$$

The larger of these values must be the maximum of f subject to the given constraint, so the constrained maximum value of f is 10.

A27. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{\sin x}{2y}$.

A: $x = \cos y + C$	B: $y = \sec x + C$	C: $y^2 = -\cos x + C$	D: $y = \dfrac{\sin x}{2} + C$	E: $y = \dfrac{\cos(x^2)}{2} + C$

Solution: This is a separable DE. We have

$$\frac{dy}{dx} = \frac{\sin x}{2y} \Rightarrow 2y\,dy = \sin x\,dx \Rightarrow \int 2y\,dy = \int \sin x\,dx \Rightarrow y^2 = -\cos x + C$$

Therefore the implicit general solution to the DE is $y^2 = -\cos x + C$.

A28. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x}{e^{2y}}$.

A: $e^{2y} = x + C$	B: $\ln\lvert y\rvert = x^2 + C$	C: $y = \ln\lvert 2x\rvert + C$	D: $y = \dfrac{\ln\lvert x\rvert}{2} + C$	E: $y = \dfrac{\ln\lvert x^2 + C\rvert}{2}$

Solution: This is another separable DE. We will need the fact that $\dfrac{e^{ky}}{k}$ is an antiderivative of e^{ky} for any constant k. (This can easily be found using the substitution $t = ky$ if you do not recognize it. Or of course in this case with $k = 2$ you would use $t = 2y$.) We get

$$\frac{dy}{dx} = \frac{x}{e^{2y}} \Rightarrow e^{2y}dy = x\,dx \Rightarrow \int e^{2y}dy = \int x\,dx \Rightarrow \frac{e^{2y}}{2} = \frac{x^2}{2} + C \Rightarrow e^{2y} = x^2 + C$$

To find the general solution $y(x)$ we need to take natural logarithms of both sides:

$$\ln e^{2y} = \ln(x^2 + C) \Rightarrow 2y = \ln(x^2 + C) \Rightarrow y = \frac{\ln(x^2 + C)}{2}$$

Notice that since $e^{2y} > 0$ for all real values of y then when $x^2 + C = e^{2y}$ we must also have $x^2 + C > 0$, so that $x^2 + C = \lvert x^2 + C\rvert$. Therefore the general solution we found can be written as $y = \dfrac{\ln\lvert x^2 + C\rvert}{2}$, which is the way it is written in the answer choices.

A29. Find the solution to $\dfrac{dy}{dx} = \dfrac{3x^2}{2y}$ subject to $y(-1) = 1$.

A: $y^2 = x^3 + 2$	B: $y^2 = x^3 - 2$	C: $y = \dfrac{3x^2}{2} + 1$	D: $y = \dfrac{3x^2}{2} - \dfrac{1}{2}$	E: $y^2 = \dfrac{x^3}{3} + \dfrac{4}{3}$

Solution: This time the separable DE has a side condition. First we find the general solution:

$$\frac{dy}{dx} = \frac{3x^2}{2y} \Rightarrow 2y\,dy = 3x^2dx \Rightarrow \int 2y\,dy = \int 3x^2dx \Rightarrow y^2 = x^3 + C$$

Now we use the side condition to find the particular solution. We know that when $x = -1$, $y = 1$ which gives

$$1^2 = (-1)^3 + C \Rightarrow 1 = -1 + C \Rightarrow C = 1 + 1 = 2$$

Therefore the (implicit) particular solution is $y^2 = x^3 + 2$.

A30. Find the solution to the differential equation $\dfrac{dy}{dx} = x^2 y$ if $y(3) = 1$.

A: $y = e^{(x^3/3)-9}$	B: $y = e^{(x^3/3)-8}$	C: $y = e^{x^3/3} + 1 - e^9$	D: $y = \ln\left(\dfrac{x^3}{3} + e - 9\right)$	E: $y = \dfrac{x^3}{3} - 8$

Solution: This DE can be considered as either a separable DE or a linear DE. Considering it as a separable DE we have $y = 0$ or

$$\frac{dy}{dx} = x^2 y \Rightarrow \frac{1}{y}\,dy = x^2 dx \Rightarrow \int \frac{1}{y}\,dy = \int x^2\,dx \Rightarrow \ln|y| = \frac{x^3}{3} + C$$

Therefore we need

$$e^{\ln|y|} = e^{\frac{x^3}{3}+C} \Rightarrow |y| = e^c e^{x^3/3} \Rightarrow y = \pm e^C e^{x^3/3}$$

Since $\pm e^C$ is an arbitrary non-zero constant then also allowing for $y = 0$ from earlier we can replace this by the arbitrary constant A (where $A = 0$ or $A = \pm e^C$). This gives the general solution as $y = Ae^{x^3/3}$.

Alternatively, considering $\dfrac{dy}{dx} = x^2 y$ as a linear DE we have $\dfrac{dy}{dx} - x^2 y = 0$. Since $-\dfrac{x^3}{3}$ is an antiderivative of $-x^2$ we get the integrating factor $u(x) = e^{-x^3/3}$ so we see that the general solution is

$$y = \frac{1}{u(x)} \int [u(x)](0)\,dx = \frac{1}{e^{-x^3/3}} \int 0\,dx = e^{x^3/3}(C) = Ce^{x^3/3}$$

(This is the same as we got before, except that the arbitrary constant has a different name. Notice that the answer was found more easily this way).

Now that we know that the general solution is $y = Ce^{x^3/3}$ (arbitrarily choosing to use C as the name of the arbitrary constant) we use the side condition to find the particular solution. We know that when $x = 3$, $y = 1$ so we get

$$1 = Ce^{3^3/3} = Ce^9 \Rightarrow C = \frac{1}{e^9} = e^{-9}$$

Therefore the particular solution is

$$y = e^{-9} e^{x^3/3} = e^{(x^3/3)-9}$$

A31. Which of the following is an integrating factor that can be used to solve the linear differential equation $y' - e^x y = 4$?

A: e^{-x}	B: e^{2x}	C: e^{-2x}	D: $e^{(-e^x)}$	E: $e^{(e^{-x})}$

Solution: The function multiplying y in the linear DE is $-e^x$ and we know that $-e^x$ is an antiderivative of $-e^x$, and an integrating factor is given by raising e to this antiderivative function. This gives the integrating factor

$$u(x) = e^{(-e^x)}$$

A32. Which one of the following is an integrating factor that can be used to solve the linear differential equation $\dfrac{dy}{dx} + \dfrac{2y}{x} = x$?

A: $\dfrac{1}{x^2}$	B: $\dfrac{1}{x}$	C: x	D: $2x$	E: x^2

Solution: This time the function multiplying y is $\dfrac{2}{x}$, so an antiderivative is $2\ln x$ which gives the integrating factor as

$$u(x) = e^{2\ln x} = e^{\ln x^2} = x^2$$

A33. Solve the linear differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = 2$.

A: $y = x^2 - x + C$	B: $y = x + C$	C: $y = 2 + \dfrac{C}{x}$	D: $y = x + \dfrac{C}{x}$	E: $y = \dfrac{1}{x} + Cx$

Solution: The function multiplying y is $\dfrac{1}{x}$ for which $\ln x$ is an antiderivative, so $u(x) = e^{\ln x} = x$ is an integrating factor which can be used to solve the DE. The general solution is given by

$$y = \frac{1}{u(x)} \int [u(x)](2)\, dx = \frac{1}{x} \int 2x\, dx = \frac{1}{x}(x^2 + C) = x + \frac{C}{x}$$

A34. Use the integrating factor $u(x) = \sec x$ to find the solution to the linear differential equation $\dfrac{dy}{dx} + y \tan x = \cos x$ subject to $y\left(\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$.

A: $y = \dfrac{2\pi}{3} - 1 - \sec x$	B: $y = \left(x - \dfrac{\pi}{3}\right) \sec x$	C: $y = \left(x + \dfrac{\pi}{3}\right) \sec x$
D: $y = \left(x + \dfrac{\pi}{3}\right) \cos x$	E: $y = \left(x - \dfrac{\pi}{3}\right) \cos x$	

Solution: The general solution is given by

$$y = \frac{1}{u(x)} \int [u(x)](\cos x)\, dx = \frac{1}{\sec x} \int (\sec x)(\cos x)\, dx = \frac{1}{1/\cos x} \int \left(\frac{1}{\cos x}\right)(\cos x)\, dx = \cos x \int dx = (\cos x)(x+C)$$

We use the side condition to solve for C. We are told that when $x = \dfrac{\pi}{3}$ we have $y = \dfrac{\pi}{3}$ as well. This gives

$$\frac{\pi}{3} = \left(\cos\frac{\pi}{3}\right)\left(\frac{\pi}{3} + C\right) = \left(\frac{1}{2}\right)\left(\frac{\pi}{3} + C\right) = \frac{\pi}{6} + \frac{C}{2}$$

Therefore $\dfrac{C}{2} = \dfrac{\pi}{3} - \dfrac{\pi}{6} = \dfrac{\pi}{6}$ and so $C = 2\left(\dfrac{\pi}{6}\right) = \dfrac{\pi}{3}$. This gives the particular solution

$$y = (\cos x)\left(x + \frac{\pi}{3}\right) = \left(x + \frac{\pi}{3}\right)\cos x$$

A35. A tank holds 1000 litres of a salt and water solution. Initially the tank contains 25 grams of salt. A solution with a concentration of 3 grams of salt per litre enters the tank at a rate of 5 litres per minute. The well-stirred solution drains from the tank at the same rate. Let y be the amount of salt in the tank at time t. Construct the mathematical model for this system.

A: $\dfrac{dy}{dt} = 5 - .003y$, $y(0) = 25$	B: $\dfrac{dy}{dt} = 3 - .005y$, $y(0) = 25$	C: $\dfrac{dy}{dt} = 15 - .005y$, $y(0) = 25$
D: $\dfrac{dy}{dt} = 3 - .001y$, $y(0) = 25$	E: $\dfrac{dy}{dt} = .015y$, $y(0) = 25$	

Solution: The volume of liquid in the tank is constant at 1000 litres. We have liquid entering the tank at a rate of 5 litres per minute, with this liquid containing 3 grams of salt per litre, so salt is entering the tank at a rate of 15 grams per minute. When there are y grams of salt in the tank, evenly spread through the 1000 litres of liquid in the tank, the concentration of salt in the liquid in the tank is $\frac{y}{1000} = .001y$ grams per litre. This liquid is leaving the tank at a rate of 5 litres per minute, so salt is leaving the tank at a rate of $(.001y)(5) = .005y$ grams per minute. The differential equation for this system simply says that the rate of change in the quantity of salt, with respect to time, is given by "rate in - rate out". Also, we are told that initially the tank contains 25 grams of salt, i.e. that at time $t = 0$ the quantity of salt in the tank is $y = 25$. Therefore the mathematical model for this situation is

$$\frac{dy}{dt} = 15 - .005y \text{ and } y(0) = 25$$

<u>PART B</u> (15 marks)

B1. [2 marks] If $f(x, y) = xe^{xy}$, find f_{xy}.

Solution: We differentiate first with respect to x and then with respect to y. We need the product rule, and the chain rule, for the first partial with respect to x.

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}(xe^{xy}) = \left[\frac{\partial}{\partial x}(x)\right](e^{xy}) + x\left[\frac{\partial}{\partial x}(e^{xy})\right] \\ &= 1(e^{xy}) + x\left[e^{xy}\left(\frac{\partial}{\partial x}(xy)\right)\right] = e^{xy} + xe^{xy}(y) = e^{xy}(1 + xy) \end{aligned}$$

And now we need both the product rule and the chain rule again when we differentiate with respect to y.

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y}(f_x(x,y)) = \frac{\partial}{\partial y}[e^{xy}(1+xy)] = \left[\frac{\partial}{\partial y}(e^{xy})\right](1+xy) + e^{xy}\left[\frac{\partial}{\partial y}(1+xy)\right] \\ &= e^{xy}\left[\frac{\partial}{\partial y}(xy)\right](1+xy) + e^{xy}(x) = e^{xy}(x)(1+xy) + xe^{xy} = e^{xy}(x + x^2y + x) \\ &= e^{xy}(2x + x^2y) \end{aligned}$$

B2. [2 marks] Find $\displaystyle\int \frac{\ln x}{x^2}\,dx$.

Solution: We need integration by parts. Letting $u = \ln x$ and $dv = \dfrac{1}{x^2}\,dx = x^{-2}dx$ we get $du = \dfrac{1}{x}\,dx$ and $v = \dfrac{x^{-1}}{-1} = -x^{-1} = -\dfrac{1}{x}$. Therefore

$$\begin{aligned} \int \frac{\ln x}{x^2}\,dx &= \int u\,dv = uv - \int v\,du = (\ln x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) \\ &= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} + \int x^{-2}dx = -\frac{\ln x}{x} + \frac{x^{-1}}{-1} + C \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C \end{aligned}$$

B3. [3 marks] Solve $\dfrac{dy}{dx} = \dfrac{1}{x\cos y}$ if $y(1) = \dfrac{\pi}{6}$.

Solution: We have a separable DE with a side condition. Separating the DE to find the general solution we get

$$\frac{dy}{dx} = \frac{1}{x\cos y} \Rightarrow \cos y\,dy = \frac{1}{x}\,dx \Rightarrow \int \cos y\,dy = \int \frac{1}{x}\,dx \Rightarrow \sin y = \ln|x| + C$$

Now we use the side condition in this implicit general solution to solve for the constant C. We know that when $x = 1$, $y = \dfrac{\pi}{6}$, so we get

$$\sin\frac{\pi}{6} = \ln 1 + C \Rightarrow \frac{1}{2} = 0 + C \Rightarrow C = \frac{1}{2}$$

This gives the implicit particular solution

$$\sin y = \ln|x| + \frac{1}{2}$$

B4. [3 marks] Solve $\dfrac{dy}{dx} - y = e^x$ if $y(1) = 2e$.

Solution: We have a linear DE. The function multiplying y is -1, which has $-x$ as an antiderivative, so we can use the integrating factor $u(x) = e^{-x}$. This gives the general solution as

$$y = \frac{1}{u(x)} \int [u(x)](e^x)\, dx = \frac{1}{e^{-x}} \int (e^{-x})(e^x)\, dx = e^x \int dx = e^x(x + C)$$

The side condition tells us that when $x = 1$, $y = 2e$ so we have

$$2e = e^1(1 + C) \Rightarrow 2e = e(1 + C) \Rightarrow C = 1$$

Therefore the particular solution is $y(x) = e^x(x + 1)$.

B5. [2 marks] A radioactive substance has a half-life of 3 days. If 200g of the substance is present initially, find the amount present after 5 days.

Solution: We know that a radioactive substance decays at a rate proportional to the quantity present, so if $y(t)$ is the number of grams of the substance present at time t days, $\frac{dy}{dt} = ky$ for some constant k. And we know that the solution to this DE is $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$. Here, we are told that $y(0) = 200$, so we have $y(t) = 200e^{kt}$. And we are told that the half-life is 3 days, i.e. that after 3 days there will only be half of the initial quantity remaining, so we know that $y(3) = 100$. This gives

$$200e^{3k} = 100 \Rightarrow e^{3k} = \frac{1}{2} \Rightarrow 3k = \ln\frac{1}{2} = -\ln 2 \Rightarrow k = -\frac{\ln 2}{3}$$

Therefore the quantity present after t days is given by $y(t) = 200e^{-t(\ln 2)/3} = 200e^{(-t/3)\ln 2}$, so the amount present after 5 days is

$$y(5) = 200e^{(-5/3)\ln 2}$$

(This could be simplified to $200(2^{-5/3}) = 25(2^{4/3})$, but that wasn't necessary.)

B6. [3 marks] The size of a certain population at any time t is given by $y(t) = y_0 e^{kt}$ for some constants y_0 and k. There are 500 in the population at time $t = 2$ and 2500 in the population at time $t = 3$.

(a) Find the values of y_0 and k.

Solution: We have $y(t) = y_0 e^{kt}$, with $y(2) = 500$ and $y(3) = 2500$. Therefore we know that $y_0 e^{2k} = 500$ and $y_0 e^{3k} = 2500$. From the first of these we get $y_0 = \dfrac{500}{e^{2k}}$, and using this in the second gives

$$\left(\frac{500}{e^{2k}}\right) e^{3k} = 2500 \Rightarrow e^{3k-2k} = \frac{2500}{500} \Rightarrow e^k = 5 \Rightarrow k = \ln 5$$

And now we use this to find the value of y_0:

$$y_0 = \frac{500}{e^{2k}} = \frac{500}{e^{2\ln 5}} = \frac{500}{e^{\ln 5^2}} = \frac{500}{5^2} = \frac{100}{5} = 20$$

(b) At what time t will the size of the population reach $100(5^{10})$?

Solution: We know that $y(t) = 20e^{t\ln 5}$ and we need to solve for t in $y(t) = 100(5^{10})$. We get

$$20e^{t\ln 5} = 100(5^{10}) \Rightarrow e^{\ln 5^t} = \frac{100(5^{10})}{20} \Rightarrow 5^t = 5(5^{10}) = 5^{11}$$

Therefore the size of the population will reach $100(5^{10})$ at time $t = 11$.