

# CARLETON UNIVERSITY

**FINAL  
EXAMINATION  
APRIL 2012**

**AUTHORIZED MEMORANDA**  
Non-programmable, non-graphic calculators

1.  $\mathcal{L}\{e^{2t} \cos(3t)\} =$  (a)  $\frac{s-2}{(s-2)^2+9}$  (b)  $\frac{s+2}{(s+2)^2+9}$  (c)  $\frac{s-3}{(s-3)^2+4}$  (d)  $\frac{s-2}{s^2+9}$   
(e) None of these
2.  $\mathcal{L}\{t \cos(2t)\} =$  (a)  $\frac{4-s^2}{(s^2+4)^2}$  (b)  $\frac{4s}{(s^2+4)^2}$  (c)  $\frac{-4s}{(s^2+4)^2}$  (d)  $\frac{s^2-4}{(s^2+4)^2}$   
(e) None of these
3.  $\mathcal{L}^{-1}\left\{\frac{s+3}{s^2-6s+18}\right\} =$  (a)  $e^{3t}[\cos(3t) + 6\sin(3t)]$  (b)  $e^{3t}[\cos(3t) + 2\sin(3t)]$   
(c)  $e^{3t}[\cos(3t) + \sin(3t)]$  (d)  $e^{-3t}[\cos(2t) + 2\sin(3t)]$  (e) None of these
4.  $\mathcal{L}^{-1}\left\{\frac{(4s+7)e^{-2s}}{s^2+s-6}\right\} =$  (a)  $u(t-2)[e^{-3t} + 3e^{2t}]$  (b)  $u(t-2)[e^{3t} + 3e^{-2t}]$   
(c)  $u(t-2)[e^{3(t-2)} + 3e^{-2(t-2)}]$  (d)  $u(t-2)[e^{-3(t-2)} + 3e^{2(t-2)}]$  (e) None of these
5. If  $y(t)$  denotes the solution of the initial-value problem

$$y'' - 2y' + 5y = 2\delta(t-2), \quad y(0) = 1, \quad y'(0) = 3,$$

- then  $Y(s) = \mathcal{L}\{y(t)\} =$  (a)  $\frac{s+3}{(s-1)^2+4}$  (b)  $\frac{s+5+2e^{-2s}}{s^2-2s+5}$  (c)  $\frac{s+1+2e^{-2s}}{s^2-2s+5}$   
(d)  $\frac{s-1+2e^{-2s}}{(s-1)^2+4}$  (e) None of these
6. If  $Y(s) = \mathcal{L}\{y(t)\} = \frac{2s+1}{s^2+s-2}$ , then  $y(t) =$  (a)  $\frac{1}{3}(e^t - e^{-2t})$  (b)  $e^{-t} + e^{2t}$   
(c)  $2e^{-\frac{1}{2}t} \cos\left(\frac{3}{2}t\right)$  (d)  $e^t + e^{-2t}$  (e) None of these
7. The general solution of the differential equation  $4x^2y'' + 8xy' + y = 0$ , valid for  $x \neq 0$ , is given by

- (a)  $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$  (b)  $|x|^{-\frac{1}{2}}(c_1 + c_2 \ln|x|)$  (c)  $c_1|x|^{-1+2\sqrt{3}} + c_2|x|^{-1-2\sqrt{3}}$   
(d)  $|x|^{-1}[c_1 \cos(2\sqrt{3} \ln|x|) + c_2 \sin(2\sqrt{3} \ln|x|)]$  (e) None of these

8. The general solution of the differential equation  $x^2y'' + 5xy' + 13y = 0$ , valid for  $x \neq 0$ , is given by

(a)  $|x|^{-\frac{5}{2}} \left[ c_1 \cos \left( \frac{3\sqrt{3}}{2} \ln |x| \right) + c_2 \sin \left( \frac{3\sqrt{3}}{2} \ln |x| \right) \right]$  (b)  $e^{-2x} [c_1 \cos(3x) + c_2 \sin(3x)]$

(c)  $c_1|x| + c_2|x|^{-5}$  (d)  $|x|^{-2} [c_1 \cos(3 \ln |x|) + c_2 \sin(3 \ln |x|)]$  (e) None of these

9. The coefficient recursion relation of the solution  $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$  of the differential equation  $x^2y'' + (x^2 - x)y' + y = 0$  is

(a)  $a_{n+1} = \frac{(n+1)a_n}{n^2}$  (b)  $a_{n+1} = \frac{n^2 a_n}{n+1}$  (c)  $a_{n+1} = \frac{a_n}{n+1}$  (d)  $a_{n+1} = \frac{-a_n}{n+1}$

(e) None of these

10. The solution of the coefficient recursion relation  $a_{n+1} = \frac{2a_n}{(n+1)^2}$ ,  $n \geq 0$ , is  $a_n =$

(a)  $\frac{2^n a_0}{(n!)^2}$  (b)  $\frac{2a_0}{(n+1)^2}$  (c)  $\frac{2^n a_0}{(n+1)!}$  (d)  $\frac{2^n a_0}{[(n+1)!]^2}$  (e) None of these

11. One solution  $y_1$  of the differential equation  $x^2y'' + (x^2 + 3x)y' + y = 0$  has the form

(a)  $\sum_{n=0}^{\infty} a_n x^n$  (b)  $\sum_{n=0}^{\infty} a_n x^{n+1}$  (c)  $\sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$  (d)  $\sum_{n=0}^{\infty} a_n x^{n-1}$  (e) None of these

12. The general solution of the differential equation  $x^2y'' + xy' + (3x^2 - 4)y = 0$ , valid for  $x > 0$ , is given by

(a)  $c_1 J_2(\sqrt{3}x) + c_2 J_{-2}(\sqrt{3}x)$  (b)  $c_1 J_{\sqrt{3}}(2x) + c_2 J_{-\sqrt{3}}(2x)$

(c)  $c_1 J_2(\sqrt{3}x) + c_2 Y_2(\sqrt{3}x)$  (d)  $c_1 J_{\sqrt{3}}(2x) + c_2 Y_{\sqrt{3}}(2x)$  (e) None of these

13. At  $x = 59$ , the Fourier sine series of  $f(x) = \left\{ \begin{array}{ll} 2, & 0 \leq x < 1 \\ 4, & 1 \leq x \leq 3 \end{array} \right\}$  on  $[0, 3]$  converges to

(a) 3 (b) -3 (c) 4 (d) -4 (e) -2

14. Let  $f(x) = \left\{ \begin{array}{ll} x, & 0 \leq x \leq 1 \\ -x, & -1 \leq x \leq 0 \end{array} \right\}$ , and  $f(x+2) = f(x)$  for all  $x$ . The Fourier series of  $f$  is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)],$$

where

(a)  $a_0 = 1, a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1], b_n = \frac{-2(-1)^n}{n\pi}, n \geq 1$

(b)  $a_0 = \frac{1}{2}, a_n = \frac{1}{n^2\pi^2}[(-1)^n - 1], b_n = \frac{(-1)^{n-1}}{n\pi}, n \geq 1$

(c)  $a_n = 0, n \geq 0, b_n = \frac{-2(-1)^n}{n\pi}, n \geq 1$

(d)  $a_n = 0, n \geq 0, b_n = \frac{2(-1)^n}{n\pi}, n \geq 1$

(e)  $a_0 = 1, a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1], b_n = 0, n \geq 1$

15. The solution of the heat equation  $u_{xx} = u_t, 0 < x < 1, t > 0$ , which satisfies the boundary conditions  $u(0, t) = u(1, t) = 0$  and the initial condition  $u(x, 0) = x$ , is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t},$$

where  $b_n =$

(a)  $\frac{-2(-1)^n}{n\pi}$     (b)  $\frac{2(-1)^n}{n\pi}$     (c)  $\frac{2(-1)^n}{n^2\pi^2}$     (d)  $\frac{-2(-1)^n}{n^2\pi^2}$     (e) None of these

16. The solution of the wave equation  $u_{xx} = \frac{1}{4}u_{tt}, 0 < x < 3, t > 0$ , which satisfies the boundary conditions  $u(0, t) = 0$  and  $u(3, t) = 0$ , and the initial conditions  $u(x, 0) = 0$  and  $u_t(x, 0) = 2 \sin(\pi x) - 3 \sin(2\pi x)$ , is

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right) \left[ a_n \cos\left(\frac{2n\pi t}{3}\right) + b_n \sin\left(\frac{2n\pi t}{3}\right) \right],$$

where

(a)  $a_1 = 2, a_2 = -3, a_n = 0$  otherwise,  $b_n = 0$  for all  $n \geq 1$

(b)  $a_3 = 2, a_6 = -3, a_n = 0$  otherwise,  $b_n = 0$  for all  $n \geq 1$

(c)  $b_1 = 2, b_2 = -3, b_n = 0$  otherwise,  $a_n = 0$  for all  $n \geq 1$

(d)  $b_3 = 2, b_6 = -3, b_n = 0$  otherwise,  $a_n = 0$  for all  $n \geq 1$

(e)  $b_3 = \frac{1}{\pi}, b_6 = \frac{-3}{4\pi}, b_n = 0$  otherwise,  $a_n = 0$  for all  $n \geq 1$

17. The solution  $u(x, y)$  of Laplace's equation  $u_{xx} + u_{yy} = 0$  within the rectangular region  $0 < x < 3, 0 < y < 2$ , subject to the boundary conditions  $u(0, y) = 0, u(3, y) = 0,$

$u(x, 0) = 3x - x^2$ ,  $u(x, 2) = 0$ , has the form

(a)  $u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{3}\right) \sin\left(\frac{n\pi x}{3}\right)$

(b)  $u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(2-y)}{3}\right] \sin\left(\frac{n\pi x}{3}\right)$

(c)  $u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)$

(d)  $u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(3-x)}{2}\right] \sin\left(\frac{n\pi y}{2}\right)$

(e)  $u(x, y) = \alpha x + \beta y + \gamma xy + \delta$

18. The bounded solution of Laplace's equation  $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$  outside the circle  $r = 3$ , which satisfies the boundary condition  $u(3, \theta) = 3 + 2\sin(2\theta) - \cos(3\theta)$ , is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where

(a)  $a_0 = 6$ ,  $a_3 = -27$ ,  $b_2 = 18$ ,  $a_n = b_n = 0$  otherwise

(b)  $a_0 = 6$ ,  $a_3 = -1$ ,  $b_2 = 2$ ,  $a_n = b_n = 0$  otherwise

(c)  $a_0 = 6$ ,  $a_3 = \frac{-1}{27}$ ,  $b_2 = \frac{2}{9}$ ,  $a_n = b_n = 0$  otherwise

(d)  $a_0 = 6$ ,  $a_2 = \frac{2}{9}$ ,  $b_3 = \frac{-1}{27}$ ,  $a_n = b_n = 0$  otherwise

(e)  $a_0 = 6$ ,  $a_2 = 18$ ,  $b_3 = -27$ ,  $a_n = b_n = 0$  otherwise

19. The differential equation  $xy'' + 2y' + xy + \lambda xy = 0$ , when placed in the Sturm-Liouville form  $[p(x)y']' - q(x)y + \lambda r(x)y = 0$ , has the weight function  $r(x) =$

(a) 1      (b)  $x$       (c)  $x^2$       (d)  $xe^{2x}$       (e) None of these

20. Given the Bessel identity  $\frac{1}{\alpha} \frac{d}{dx} [x^\nu J_\nu(\alpha x)] = x^\nu J_{\nu-1}(\alpha x)$ ,  $\nu > 0$ ,  $\alpha \neq 0$ ,

$\int_0^2 x^4 J_1(3x) dx =$       (a)  $\frac{16}{9}[3J_2(6) - J_3(6)]$       (b)  $16J_1(6)$       (c)  $\frac{32}{5}J_2(6)$

(d)  $16[J_2(6) - J_3(6)]$       (e) None of these

21. The eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < 2, \quad y(0) = 0, \quad y'(2) = 0,$$

are

(a)  $\lambda_n = \frac{n\pi}{2}, y_n = B_n \sin\left(\frac{n\pi x}{2}\right), n \geq 1$

(b)  $\lambda_n = \frac{n\pi}{2}, y_n = A_n \cos\left(\frac{n\pi x}{2}\right), n \geq 0$

(c)  $\lambda_n = \frac{(2n+1)^2\pi^2}{16}, y_n = B_n \sin\left[\frac{(2n+1)\pi x}{4}\right], n \geq 0$

(d)  $\lambda_n = \frac{(2n+1)^2\pi^2}{16}, y_n = A_n \cos\left[\frac{(2n+1)\pi x}{4}\right], n \geq 0$

(e)  $\lambda_n = \frac{(2n+1)^2\pi^2}{4}, y_n = A_n \cos\left[\frac{(2n+1)\pi x}{2}\right], n \geq 0$

22.  $\mathcal{F}\{e^{-2ix-|x-3|}\} =$  (a)  $\frac{2e^{-3i(\lambda-2)}}{1+(\lambda-2)^2}$  (b)  $\frac{2e^{3i(\lambda-2)}}{1+(\lambda-2)^2}$  (c)  $e\frac{2e^{-3i(\lambda+2)}}{1+(\lambda+2)^2}$

(d)  $\frac{2e^{3i(\lambda+2)}}{1+(\lambda+2)^2}$  (e) None of these

23.  $\mathcal{F}\{2xe^{-x^2}\} =$  (a)  $i\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}$  (b)  $2\lambda e^{-\lambda^2}$  (c)  $\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$  (d)  $-i\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}$

(e) None of these

24.  $\mathcal{F}^{-1}\left\{\frac{e^{-3i\lambda}}{1+(\lambda+2)^2}\right\} =$  (a)  $\frac{1}{2}e^{2i(x+3)-|x+3|}$  (b)  $\frac{1}{2}e^{-2i(x+3)-|x+3|}$

(c)  $\frac{1}{2}e^{-2i(x-3)-|x-3|}$  (d)  $\frac{1}{2}e^{2i(x-3)-|x-3|}$  (e) None of these

25.  $\mathcal{F}^{-1}\{\lambda e^{-|\lambda|}\} =$  (a)  $\frac{-i}{\pi(1+x^2)^2}$  (b)  $\frac{i}{\pi(1+x^2)^2}$  (c)  $\frac{-2ix}{\pi(1+x^2)^2}$

(d)  $\frac{2ix}{\pi(1+x^2)^2}$  (e) None of these

**Table of Laplace Transforms**

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt, \quad s > 0$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, \quad p > -1$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right), \quad \alpha > 0$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad s > a \geq 0$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n} F(s), \quad n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(x) dx$$

$$\mathcal{L}\left\{\int_0^t f(x) dx\right\} = \frac{1}{s} F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x) dx\right\} = F(s)G(s), \text{ where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt \text{ if } f \text{ is periodic with period } T$$

### Summary of Fourier Series

1. The Fourier series of a  $2L$ -periodic function  $f$  is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1,$$

where  $\alpha$  is any real number. If  $f$  is an odd function, then

$$a_n = 0 \quad \text{and} \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

If  $f$  is an even function, then

$$b_n = 0 \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$

2. The Fourier series of a function  $f$  defined on  $[a, b]$  with  $b - a = 2L$  is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L} \int_a^b f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L} \int_a^b f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

If the  $2L$ -periodic extension  $\tilde{f}$  of  $f$  to  $\mathbb{R}$  is an odd function, then  $a_n = 0$ , and if  $\tilde{f}$  is an even function, then  $b_n = 0$ .

3. The Fourier sine series of a function  $f$  defined on  $[0, L]$  is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

4. The Fourier cosine series of a function  $f$  defined on  $[0, L]$  is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$

## Table of Fourier Transforms

$$\mathcal{F}\{f(x)\} = \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx$$

$$\mathcal{F}^{-1}\{F(\lambda)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x} d\lambda$$

$$\mathcal{F}\{u(x-a) - u(x-b)\} = \frac{1}{i\lambda} (e^{i\lambda b} - e^{i\lambda a}), \quad a < b$$

$$\mathcal{F}\{u(x+b) - u(x-b)\} = \frac{1}{i\lambda} (e^{i\lambda b} - e^{-i\lambda b}) = \frac{2}{\lambda} \sin(\lambda b)$$

$$\mathcal{F}\{e^{-|x|}\} = \frac{2}{1 + \lambda^2}$$

$$\mathcal{F}\{e^{iax} f(x)\} = \widehat{f}(\lambda + a)$$

$$\mathcal{F}\{f(x-a)\} = e^{i\lambda a} \widehat{f}(\lambda)$$

$$\mathcal{F}\{f'(x)\} = -i\lambda \widehat{f}(\lambda)$$

$$\mathcal{F}\{xf(x)\} = -i \frac{d\widehat{f}}{d\lambda}$$

$$\mathcal{F}\{e^{-tx^2}\} = \sqrt{\frac{\pi}{t}} e^{-\frac{\lambda^2}{4t}}, \quad t > 0$$

$$\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|} \widehat{f}\left(\frac{\lambda}{\alpha}\right), \quad \alpha \neq 0$$

$$\mathcal{F}\{(f * g)(x)\} \equiv \mathcal{F}\left\{\int_{-\infty}^{\infty} f(s)g(x-s) ds\right\} = \widehat{f}(\lambda)\widehat{g}(\lambda), \quad \text{where } \widehat{g}(\lambda) = \mathcal{F}\{g(x)\}$$

$$\mathcal{F}\{\delta(x-a)\} = e^{i\lambda a}$$