SFU faculty of science
department of mathematics

Midterm 1 v1

MATH 150 D100 Fall 2013

Instructor: Dr. C. Drakes

Sept 30, 2013, 8:30-9:20 am

Name: Solutions. (please print)
family name *given name*

SFU ID: @sfu.ca
student number *SFU-email*

Signature:

Instructions:

1. Do not open this booklet until told to do so.
2. Write your name above in block letters. Write your SFU student number and email ID on the line provided for it.
3. Write your answer in the space provided below the question. If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
5. This exam has 6 questions on 6 pages (not including this cover page). Once the exam begins please check to make sure your exam is complete.
6. **No** calculators, books, papers, or electronic devices shall be within the reach of a student during the examination. Leave answers in "calculator ready" expressions: such as $3 + \ln 7$ or $e^{\sqrt{2}}$.
7. **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**
8. Good Luck!

Question	Maximum	Score
1	12	
2	15	
3	8	
4	6	
5	6	
6	3	
Total	50	

1. Let f be the function defined by

$$f(x) = \frac{-x}{x+1}$$

[3] (a) Find $f \circ f(x)$.

$$f(f(x)) = f\left(\frac{-x}{x+1}\right) = \frac{-\left(\frac{-x}{x+1}\right)}{\frac{-x}{x+1} + 1} = \frac{x}{x+1} \div \left(\frac{-x}{x+1} + \frac{x+1}{x+1}\right)$$

$$= \frac{x}{\cancel{x+1}} \times \frac{\cancel{x+1}}{1} = x$$

[3] (b) Find the domain and range of f.

Domain = Range Since $f(x) = f^{-1}(x)$

$D: \{x \in \mathbb{R} \mid x \neq -1\}$

$R: \{y \in \mathbb{R} \mid y \neq -1\}$

[3] (c) Give the inverse of f.

$f^{-1}(x) = f(x) = \frac{-x}{x+1}$ since $f(f(x)) = x$

[3] (d) Use **limits** to find the vertical and horizontal asymptotes to the graph of f

V.A.

$$\lim_{x \to -1^-} \frac{-x}{x+1} = -\infty$$

$$\lim_{x \to -1^+} \frac{-x}{x+1} = \infty$$

H.A

$$\lim_{x \to \pm\infty} \frac{-x}{x+1} = \lim_{x \to \infty} \frac{x(-1)}{x(1 + \cancel{1/x})} = -1$$

2. Compute the following limits. For full credit you must **justify** your answers.

[3] (a) $\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t^2+t}\right)$

$$= \lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t(t+1)}\right)$$

$$= \lim_{t\to 0}\left(\frac{t+1-1}{t(t+1)}\right)$$

$$= \lim_{t\to 0}\frac{1}{t+1} = 1$$

[3] (b) $\lim_{v\to 4^+}\frac{v-4}{|4-v|}$

$$|4-v| = \begin{cases} 4-v & \text{if } 4-v \geq 0 \\ -(4-v) & \text{if } 4-v < 0 \end{cases}$$

$$= \begin{cases} 4-v & \text{if } v \leq 4 \\ v-4 & \text{if } v > 4 \end{cases}$$

$$\Rightarrow \lim_{v\to 4^+}\frac{\cancel{v-4}}{\cancel{v-4}}$$

$$= 1$$

[3] (c) $\lim_{x\to 2}\frac{\sqrt{4x+1}-3}{x-2}$ $\frac{(\sqrt{4x+1}+3)}{(\sqrt{4x+1}+3)}$

$$= \lim_{x\to 2}\frac{4x+1-9}{(x-2)(\sqrt{4x+1}+3)}$$

$$= \lim_{x\to 2}\frac{4x-8}{(x-2)(\sqrt{4x+1}+3)}$$

$$= \lim_{x\to 2}\frac{4\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{4x+1}+3)}$$

$$= \frac{4}{\sqrt{9}+3} = \frac{4}{6} = 2/3$$

[3] (d) $\lim_{x\to 0}\left(x^4\cos\left(\frac{3}{x}\right)\right)$ $\longrightarrow$ direct sub doesn't work since $\lim_{x\to 0} \cos\left(\frac{3}{x}\right)$ is undefined.

But we know $$-1 \le \cos\left(3/x\right) \le 1$$

$$\Rightarrow -x^4 \le x^4\cos\left(3/x\right) \le x^4$$

As $x \to 0$ $-x^4 \to 0$ and $x^4 \to 0$

$\therefore$ By Squeeze theorem $x^4\cos\left(3/x\right) \to 0$

[3] (e) $\lim_{x\to -\infty} \frac{\sqrt{x^2-9}}{2x-6}$

Note

$x < 0$

$\Rightarrow -x > 0$

$$\lim_{x\to -\infty} \frac{\sqrt{x^2\left(1 - 9/x^2\right)}}{x\left(2 - 6/x\right)}$$

$$= \lim_{x\to -\infty} \frac{-\cancel{x}\sqrt{1 - \cancel{9/x^2}^{\,0}}}{\cancel{x}\left(2 - \cancel{6/x}\right)^{0}}$$ since $-x = {}^{+}\sqrt{x^2}$

$$= \frac{-\sqrt{1}}{2} = -1/2$$

[3] **3.** (a) Define what it means for a function to be continuous at a point $x = a$

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

[5] (b) Find the values of d and c that make the function continuous everywhere:

$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ dx^2 - cx + 3 & \text{if } 2 \le x < 3 \\ 2x - d + c & \text{if } x \ge 3 \end{cases}$$

- We know that $g(x)$ is continuous on each branch since polynomial & rational functions are continuous on their domains.

- @ $x = 2$

$$\left.\begin{aligned} &g(2) = 4d - 2c + 3 \\ &\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4 \\ &\lim_{x \to 2^+} g(x) = g(2) = 4d - 2c + 3 \end{aligned}\right\}$$

For continuity
$4d - 2c + 3 = 4$
$\Rightarrow 4d - 2c = 1$

- @ $x = 3$

$$\left.\begin{aligned} &g(3) = 6 - d + c = \lim_{x \to 3^+} g(x) \\ &\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} dx^2 - cx + 3 = 9d - 3c + 3 \end{aligned}\right\}$$

For continuity
$9d - 3c + 3 = 6 - d + c$
$\Rightarrow 10d - 4c = 3$

∴ Solving system: $(4d - 2c = 1) \times 2$
$10d - 4d = 3$

$\Rightarrow$ $8d - 4c = 2$
$-\ 10d - 4c = 3$

$-2d = -1 \Rightarrow d = 1/2 \Rightarrow c = 1/2$

[2] **4.** (a) State the **Intermediate Value Theorem** clearly identifying all the hypotheses and the conclusion.

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ & $f(b)$ where $f(a) \neq f(b)$.

Then there exists a number c in (a, b) such that $f(c) = N$.

[4] (b) Use the Intermediate Value Theorem to show that

$$e^x = 3 - 2x$$

for some x in the interval $[0, 1]$

- Let $f(x) = e^x - 3 + 2x$

  $N = 0$.

  ~~We want to show that~~

- $f(0) = e^0 - 3 + 0$

  $= 1 - 3 = -2 < 0$

opposite signs.

$f(1) = e^1 - 3 + 2$

$= e + 2 - 3 > 0$

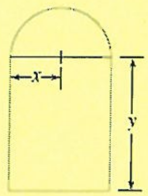
approx 2.718.

$\Rightarrow N = 0$ is between $f(0)$ & $f(1)$

$\therefore$ By IVT there exists a number c in $(0, 1)$ such that $f(c) = N = 0$ $\Rightarrow e^c - 3 + 2c = 0$

$\Rightarrow e^c = 3 - 2c$

[6] **5.** A Norman window has the shape of a rectangle surmounted by a semicircle (see figure). Suppose a Norman window for a church is to have a **perimeter** of 28m; find a function in the variable x for the **area** of the window.



Perimeter = 28

$\Rightarrow 2y + 2x + \frac{2\pi x}{2} = 28$

$\Rightarrow 2y = 28 - 2x - \pi x$

$y = 14 - x - \pi/_2\, x$

$\text{Area} = 2xy + \frac{\pi x^2}{2}$

$A(x) = 2x\left(14 - x - \pi/_2\, x\right) + \frac{\pi}{2}x^2$

$= 28x - 2x^2 - \pi x^2 + \pi/_2\, x^2$

$A(x) = 28x - \left(2 + \pi/_2\right)x^2$

[3] **6.** If $g \circ f$ is defined at $x = a$, must $f \circ g$ be defined at $x = a$? If yes explain why. If no, give an example to show why it is not true.

No.

Let $a = -1$, $g(x) = \sqrt{x}$ $f(x) = x + 7$

$(g \circ f)(a) = g(f(a))$
$= g(-1+7)$
$= g(6)$
$= \sqrt{6}$
Defined.

$(f \circ g)(a) = f(g(a))$
$= f(\sqrt{-1})$
DNE.