University of Ottawa

Department of Mathematics and Statistics

MAT 1302D: Mathematical Methods II
Instructor: Catalin Rada

Midterm 1 – January 30, 2014

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(1) You have 80 minutes to complete this exam.
(2) Show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(3) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
provided.
(4) No notes, books, calculators or scrap paper are allowed.
(5) The final page of the exam may be used for scrap work.
(6) Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	4	2	4	2	5	20
Score							

1. **(3 points)** Determine if the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$$

is consistent or inconsistent. If it is consistent, determine its general solution.

Solution:

The augmented matrix associated to the system corresponding to the above vector equation is

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 2 & -4 & 4 & -6 & 8 \end{array}\right]$$

It is row reduced as follows:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 2 & -4 & 4 & -6 & 8 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - 2R_1}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 6 & -12 & 6 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{3} R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The rightmost column is not a pivot column, so the system is consistent. Moreover, x_2 and x_4 are free variables, and x_1, x_3 are basic variables. One obtains $\begin{cases} x_1 = 2 - x_4 + 2x_2 \\ x_2 = \text{free} \\ x_3 = 2x_4 + 1 \\ x_4 = \text{free} \end{cases}$

2. **(4 points)** Is the following linear system consistent? If so, find the general solution and write it in vector parametric form.

$$\begin{array}{rcrcrcrcr} 2x & + & 4y & - & 4z & + & 6w & = & 4 \\ 2x & + & 4y & - & 3z & + & 4w & = & 5 \\ 5x & + & 10y & - & 8z & + & 11w & = & 12 \end{array}$$

Solution: The augmented matrix of the above system of linear equations is

$$A = \left[\begin{array}{cccc|c} 2 & 4 & -4 & 6 & 4 \\ 2 & 4 & -3 & 4 & 5 \\ 5 & 10 & -8 & 11 & 12 \end{array}\right]$$

It is row reduced as follows:

$$A = \left[\begin{array}{cccc|c} 2 & 4 & -4 & 6 & 4 \\ 2 & 4 & -3 & 4 & 5 \\ 5 & 10 & -8 & 11 & 12 \end{array}\right] \xrightarrow{R_1 \to 2^{-1}R_1} \left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 2 \\ 2 & 4 & -3 & 4 & 5 \\ 5 & 10 & -8 & 11 & 12 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - 5R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The rightmost column is not a pivot column, therefore the system is consistent, and moreover: x, z are basic variables and y, w are free. It follows that $\begin{cases} x = -2y + w + 4 \\ y = \text{free} \\ z = 2w + 1 \\ w = \text{free} \end{cases}$

and the vector parametric form of the solution is:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2y + w + 4 \\ y \\ 2w + 1 \\ w \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

3. **(2 points)** Find all of the value(s) of h for which the following system has a unique solution:

$$\begin{array}{rcrcl} x & + & y & = & 4 \\ -8x & + & hy & = & 6 \end{array}$$

Solution:
The augmented matrix of the above system of linear equations is

$$A = \left[\begin{array}{rr|r} 1 & 1 & 4 \\ -8 & h & 6 \end{array}\right]$$

Add 8 Rows 1 to Row 2 and obtain $\left[\begin{array}{cr|r} 1 & 1 & 4 \\ 0 & h+8 & 38 \end{array}\right]$

To get a unique solution, one must impose that $h \neq -8$.

4. **(4 points)** Suppose that

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix}.$$ Is the vector $b = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 4 \end{bmatrix}$ in $Span\{v_1, v_2, v_3\}$? Justify your answer!

Solution:

Note that the augmented matrix of the corresponding system can be row reduced as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 4 & 4 \end{array}\right] \xrightarrow{R_4 \to 2^{-1} R_4}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{array}\right] \xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 - R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & -2 & -6 \end{array}\right] \xrightarrow{R_4 \to R_4 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to (-1) R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - 2R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Because the system is consistent, the answer is positive.

5. **(2 points)** Give an example of a 3×3 matrix that is NOT in Reduced echelon form, and whose columns span $\mathbf{R}^3$. Justify (e.g., using row operations on your matrix) your example!

Solution:
An example:

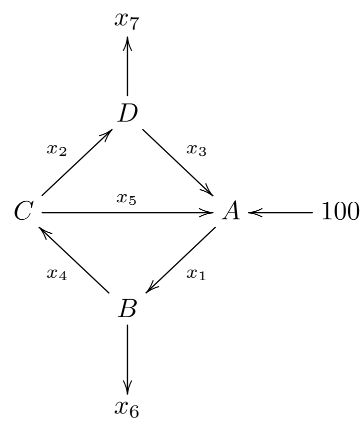
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Row echelon form: $(R_2 \rightarrow R_2 - R_1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

contains a pivot position in each row!

6. **(5 points)** The traffic flow in a a blue village is represented by the diagram below. The arrows indicate the directions of one-way traffic. Numbers and variables indicate the number of cars per hour in the corresponding direction.



(a) **(3 points)** Write down a linear system describing the traffic flow. Do <u>not</u> solve the linear system at this stage.

Solution:

Total: $100 = x_6 + x_7$
A : $100 + x_3 + x_5 = x_1$
B : $x_1 = x_4 + x_6$
C : $x_4 = x_2 + x_5$
D : $x_2 = x_3 + x_7$

(b) **(2 points)** The reduced echelon form of the augmented matrix of the above linear system is the following matrix:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & 1 & 100 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Suppose that due to road closures, the maximum number of cars travelling from B to C is limited to 50 cars per hour. What is the maximum number of cars per hour that can travel from A to B? Justify your answer.

Solution: From the above matrix we have $x_1 = x_4 - x_7 + 100$. If $x_4 \leq 50$ then $x_1 \leq 150$ (since $x_7 \geq 0$).

This page is for scrap work and is intentionally left blank.