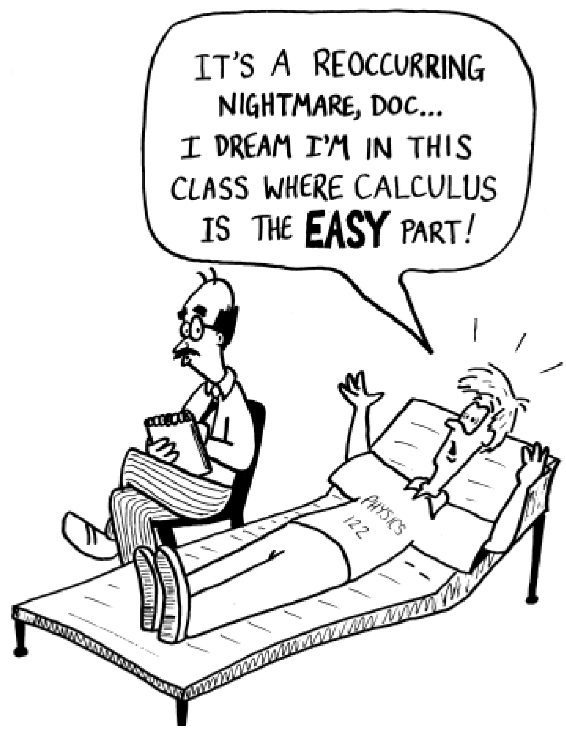
# The Derivative Part 2

MAT 1300 C

Winter 2014



# 1 The Chain Rule

**The Chain Rule:** Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

**The General Power Rule:** Let $g(x)$ be a differentiable function and let $h(x) = [g(x)]^n$. Then,

$$h'(x) = n[g(x)]^{n-1}g'(x).$$

Ex: Given $r(z) = (z^2 - 3z)^5$, find $r'(z)$.

Ex: Given $f(x) = \sqrt{2x+3}$, find $f'(x)$.

Ex: Given that $h(x) = x(5x+1)^3$, find $h'(x)$.

Ex: Find $g'(2)$ given that

$$g(x) = \frac{(x^2+4)^{1/3}}{x}.$$

Ex: Find the equation of the tangent line to

$$f(x) = \frac{5}{(2x-1)^7}$$

at the point $x = 1$.

Ex: The cost of producing $x$ units of a product is given by $C(x) = \sqrt{3x^2 + 400}$, where $C(x)$ is the cost in dollars. Find the marginal cost for producing 20 units.

# 2 Implicit Differentiation

Many relationships can be expressed explicitly in the form $y = f(x)$, but some can only be written implicitly as a mix of $x$'s and $y$'s on either side of the equality sign.

Ex:

$$x^2 - 5y^4 + 3y + 2x = 10$$

Here, *implicitly* $y$ is a function of $x$. If we want to know $\frac{dy}{dx}$ how can we find it?

A new rule for equations:

"You can take the derivative of the entirety of both sides of a valid equation with respect to the same variable and you will get a new equation that is valid."

In this sense "taking the derivative with respect to $x$" is an operation we can perform on an equation (just like "multiplying both sides by 7") that preserves the validity of the equation.

If we are taking the derivative of some term with respect to $x$ which contains a function of some other variable, say $h(y)$ then we use the chain rule to obtain $h'(y)\frac{dy}{dx}$ in the derivative.

**Implicit Differentiation of $y^n$**

If $y = f(x)$, then

$$\frac{d}{dx}(y^n) = n.y^{n-1}\frac{dy}{dx}.$$

Ex: Find $\frac{dy}{dx}$ as a function of $y$ and $x$ given that $[x - y^2] = 0$ at the point (4,2).

Ex: Find $\frac{dy}{dx}$ as a function of $y$ and $x$ given that

$$x^2 - 5y^4 + 3y + 2x = 10.$$

To find the derivative of a function, which is defined implicitly, at a specific point we need to use implicit differentiation first, but then we can substitute in the values from the point *before we simplify*.

Ex: Find $\frac{dy}{dx}$, when $x = 2$ if

$$y^3 - 10xy = 27 - 5x^2y.$$

Ex: Find the equation of the tangent line to the curve $y^2 + xy - x^2 = 5$ at the point $(0, \sqrt{5})$.

Ex: Find $\frac{dy}{dx}$ for $xy^3 = 54$ and evaluate it at the point (2,3).

# 3 Derivatives of Logarithmic Functions

Fact:

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}.$$

In general,

$$\begin{aligned} \frac{d}{dx}[\log_b(x)] &= \frac{d}{dx}\left[\frac{\ln(x)}{\ln(b)}\right] \\ &= \frac{1}{\ln(b)}\frac{1}{x}. \end{aligned}$$

Also by the chain rule, for any positive-valued function, $f(x)$,

$$\frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)}f'(x).$$

Ex: Compute $f'(x)$ given that

$$f(x) = 5\ln(3x + 7).$$

Ex: Compute $\frac{dy}{dx}$ given that

$$y = x \ln\left(\frac{x^2 - 5x}{2x + 1}\right).$$

# 4 Derivatives of Exponential Functions

Goal: Find the derivative of $e^x$ using implicit differentiation.

So,

$$\frac{d}{dx}[e^x] = e^x.$$

By the chain rule, for any function, $f(x)$,

$$\frac{d}{dx}[e^{f(x)}] = e^{f(x)}f'(x).$$

Ex: Find $f'(x)$ given that

$$f(x) = x^2 e^{3x+1}.$$

Ex: Find $f'(x)$ given that

$$f(x) = e^{3x+1} \ln(3x).$$

Ex: For $y = x^2 - 3\ln(x)$ find the slope and the equation of the tangent line to $f(x)$ at the point $x = 2$.

In general,

$$\begin{aligned}\frac{d}{dx}[a^x] &= \frac{d}{dx}[(e^{\ln(a)})^x] \\ &= \frac{d}{dx}[e^{\ln(a)x}] \\ &= e^{\ln(a)x}(\ln(a)) \\ &= \ln(a)a^x.\end{aligned}$$

We have,

$$f(x) = a^x \qquad f'(x) = a^x.\ln(a)$$

$$y = \left(a^{f(x)}\right) \qquad y' = f'(x).a^{f(x)}.\ln(a).$$

Ex: Find $f'(x)$ given that $f(x) = 6^x$.

Ex: Find $f'(x)$ given that $f(x) = e^x 3^{2x+1}$.