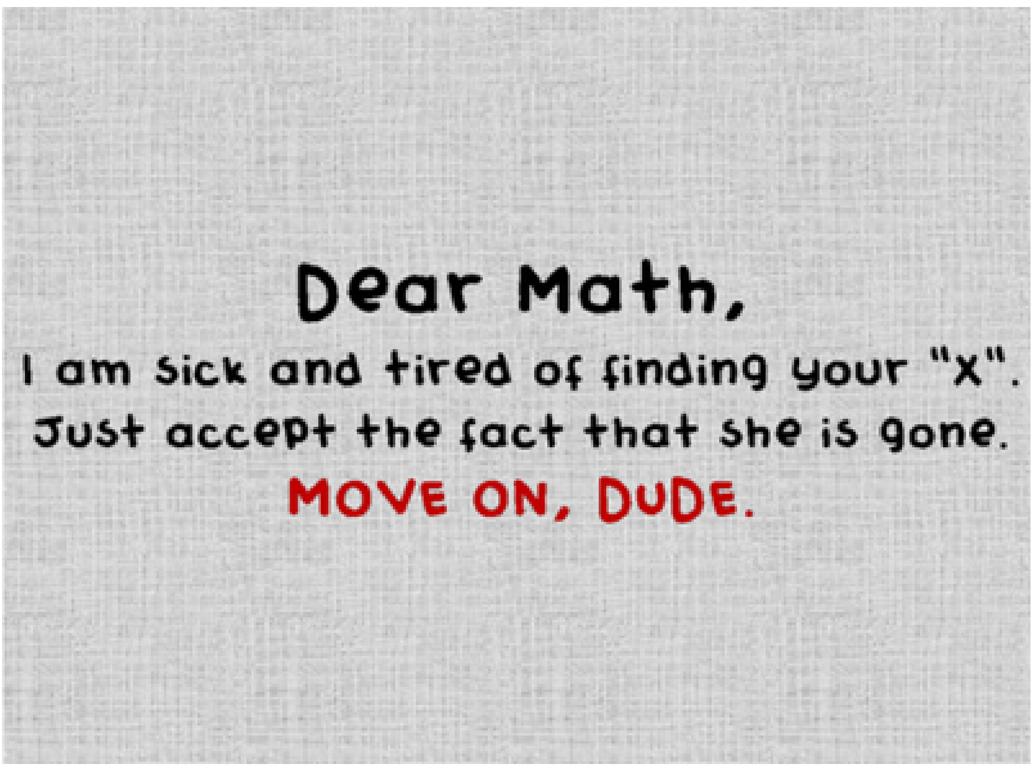
The Derivative

MAT 1300 C

Winter 2014



1 Approximating with Linear Functions

Ex: The sales of long distance phone time in minutes, q, depends on the price per minute in cents, p, as follows.

$$q = f(p) = 5000 - 30p^2$$

If the current rate is \$0.10 per minute, find a "good" linear model for the demand so we can estimate how small changes in the price will affect the demand.

We call the line which passes through the point (10,2000) and whose slope is the same as the "slope" of the curve the *tangent line* to the curve at (10,2000).

Ex: a) Calculate the slope of the tangent line for the preceding example.

b) Find the equation of the tangent line.

c) Use the tangent line to estimate the demand if the we raise the price per minute to \$0.11. How does this compare with the true demand at this price?

2 Definition of the Derivative

Let $f(x)$ be a function. Then we define **the derivative of** f **at** x, denoted $f'(x)$, as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

if that limit exists, and if it does we say that f is **differentiable** at x.

Notation: We will denote this a few different ways. If $y = f(x)$, then all of the following denote the derivative:

$$f'(x),\ \frac{df}{dx},\ \frac{\mathrm{d}}{dx}(f(x)),\ y',\ \frac{dy}{dx}.$$

Examples:

1. Find the derivative of $f(x) = \frac{1}{2}x + 3$ at $x = 2$.

   We calculate:

$$\begin{aligned} f'(2) &= \lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}(2+\Delta x) + 3 - (\frac{1}{2}2 + 3)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}\Delta x}{\Delta x} = \frac{1}{2} \end{aligned}$$

   So the slope of the tangent line is $m = \frac{1}{2}$. This makes sense because $y = \frac{1}{2}x + 3$ is a line, and the tangent line is the line itself.

2. Find the slope for the tangent line for $g(x) = 1 - x^2$ at x (unspecified).

   We have

$$\begin{aligned} g'(x) &= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1 - (x+\Delta x)^2 - (1 - x^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1 - (x^2 + 2x\Delta x + \Delta x^2) - (1 - x^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-2x\Delta x - \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (-2x - \Delta x) \\ &= -2x \end{aligned}$$

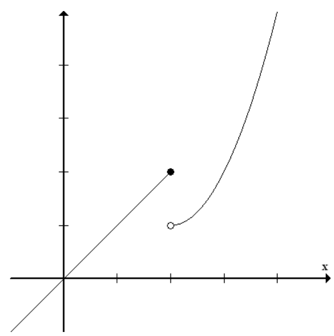
   So the slope of the tangent line at x is $-2x$.

3. Find the **equation** of the tangent line to $f(x) = \frac{1}{x}$ at the point $(1, 1)$.

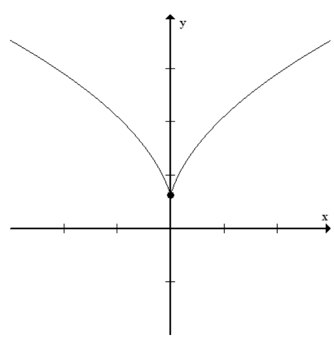
4. Find $f'(x)$, where $f(x) = \sqrt{x+1}$.

<u>Notes:</u>

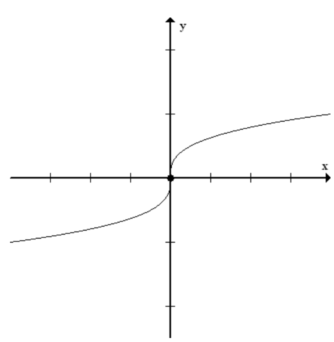
- of discontinuity,



- with corners,



- or with vertical tangents.



3 Basic Rules for Differentiation

Constant Rule: Let c be a constant function (a function whose value is c regardless of x). Then

$$\frac{d}{dx}[c] = 0.$$

Ex: Given $f(x) = -3.9$, find $f'(x)$.

Power Rule: Let n be any real number. Then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Ex: Given $h(z) = z^{0.6}$, find $\frac{dh}{dz}$.

Constant Multiple Rule: Let c be any constant and $f(x)$ any function. Then

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Ex: Given $g(y) = 7y^{-3}$, find $g'(y)$.

Sum Rule: Let $f(x)$ and $g(x)$ be any functions. Then

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x).$$

Examples:

1. Find the derivative of $f(t) = t^{2/3} + 3t - 3$.

   solution: We use a combination of the sum rule, constant multiple rule, and power rule:

$$\begin{aligned}\frac{\mathrm{d}}{dt}(t^{2/3}+3t-3) &= \frac{2}{3}t^{2/3-1} + 3t^{1-1} + 0 \\ &= \frac{2}{3}t^{-1/3} + 3\end{aligned}$$

2. Find the derivative of $f(x) = \frac{4}{x^3}$.

3. Given $f(x) = 2x^{4/5} + 7$, find $f'(x)$.

4 The Product Rule

The Product Rule: Let $f(x)$ and $g(x)$ be differentiable functions. Then,

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

1. Differentiate $f(x) = (x^2 + 1)(2x + 5)$ (differentiate means "find the derivative of").

2. Given $h(x) = -3x^{5/2}(1 - 2x^4)$, use the product rule to find $f'(x)$.

5 The Quotient Rule

The Quotient Rule: Let $f(x)$ and $g(x)$ be differentiable functions where $g(x) \neq 0$. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

1. Find the derivative of $g(x) = \left(\frac{1}{x} + 3\right)\left(\frac{1}{x^2} - 4\right)$.

2. Find the derivative of $h(x) = \frac{\sqrt{x}}{x+1}$.

3. Let $f(x) = \frac{x^2-x-3}{x^2+1}$
   (a) Find $f'(x)$.
   (b) Find the equation of the tangent line to f at $(1, -\frac{3}{2})$.

6 Higher-Order Derivatives

Notice, for any function, $f(x)$, that $f'(x)$ is also a function of x. Hence, we can take the derivative of $f'(x)$ with respect to x. The **second order derivative**, or just **second derivative** of $f(x)$, if it exists, is

$$f''(x) = \frac{\mathrm{d}}{dx} f'(x)$$

The **third derivative** is

$$f'''(x) = \frac{\mathrm{d}}{dx} f''(x)$$

and so on. For higher order derivatives we use superscripts: $f^{(4)}$ = fourth derivative etc.

Example: Find $f''(x)$ given that

$$f(x) = (x^3 + 1)(2x^9 + 5).$$