

Name: MODEL ANSWERS

Student Number: \_\_\_\_\_

**WILFRID LAURIER UNIVERSITY**

Waterloo, Ontario

Mathematics 235 – Introduction to Game Theory

Test I – October 11, 2011

Instructor: *Dr. M. Kilgour*

Time Allowed: *80 minutes*

Total Value: *80 marks*

Number of Pages: *6 plus cover page*

**Instructions:**

*Non-programmable, non-graphing calculators are permitted. No other aids are allowed.*

*Check that your test paper has no missing, blank, or illegible pages.*

*Answer in the spaces provided. Please note that questions are printed on both sides of the test pages.*

*Show all your work. Insufficient justification will result in a loss of marks.*

[8 marks] 1. Let  $W_1 = \{a, b, c, d\}$ . The relation  $B$  is defined on  $W_1$  by  $x B y$  if and only if the table on the right contains an  $X$  in the cell  $(x, y)$ . Answer the following questions about the relation  $B$ . Justify your answers.

	$y = a$	$y = b$	$y = c$	$y = d$
$x = a$	X	X		
$x = b$		X	X	
$x = c$		X		X
$x = d$	X		X	

(a) Is  $B$  reflexive?

(2) No, because  $\neg(c B c)$   
(or  $\neg(d B d)$ )

(b) Is  $B$  antisymmetric?

(2) No, because  $d B a$  and  $c B d$   
(or  $c B b$  and  $b B c$ )

(c) Is  $B$  symmetric?

(2) No, because  $d B a$  and  $\neg(a B d)$   
(or  $a B b$  and  $\neg(b B a)$ )

(d) Is  $B$  transitive?

(2) No, because  $a B b$  and  $b B c$  but  $\neg(a B c)$   
(Many other examples)

[6 marks] 2. (a) Prove that if a relation  $R$  on a non-empty set,  $W$ , is complete, then  $R$  is also reflexive.

If  $R$  is complete (see def'n below) and  $y = x$

then  $x R y$  or  $y R x$

i.e.  $x R x$  or  $x R x$

Therefore  $x R x$  for any  $x \in W$

$\therefore R$  is reflexive

(b) Suppose that  $D$  is a relation on the set  $W_1$  (see #1) that is reflexive, symmetric, and anti-symmetric. Either construct a table defining the relation  $D$ , or show that no such relation can exist.

(3) Only relation that satisfies all requirements

	$a$	$b$	$c$	$d$
$a$	X			
$b$		X		
$c$			X	
$d$				X

If  $B$  is a relation on  $W$ , then

$B$  is reflexive iff, for all  $x \in W$ ,  $x B x$ .

$B$  is irreflexive iff, for all  $x \in W$ ,  $x \neg B x$ .

$B$  is symmetric iff, for all  $x \in W$ , and  $y \in W$ , if  $x B y$ , then  $y B x$  also.

$B$  is asymmetric iff, for all  $x \in W$ , and  $y \in W$ , if  $x B y$ , then  $\neg(y B x)$ .

$B$  is antisymmetric iff, for all  $x \in W$  and  $y \in W$  such that  $x \neq y$ , if  $x B y$  then  $\neg(y B x)$ .

$B$  is transitive iff, for all  $x \in W$ , and  $y \in W$  and  $z \in W$  if  $x B y$  and  $y B z$  then  $x B z$  also.

$B$  is complete iff, for every  $x \in W$  and  $y \in W$ , either  $x B y$  or  $y B x$ .

$B$  is weakly complete iff, for every  $x \in W$  and  $y \in W$  such that  $x \neq y$ , either  $x B y$  or  $y B x$ .

- [6 marks] 3. Recall that a weak preference relation,  $\succeq$ , on a set  $W$  must be reflexive, transitive, and complete. The corresponding strict preference relation,  $\succ$ , on  $W$  is defined by

$$x \succ y \text{ if and only if } x \succeq y \text{ and } \neg(y \succeq x)$$

for any  $x, y \in W$ . Suppose that  $r, s, t \in W$  satisfy  $r \succeq s$  and  $s \succ t$ . Prove that  $r \succ t$ .

[Hint: Carry out the following steps: (i) Prove that  $r \succeq t$ . (ii) Show that the assumption that  $t \succeq r$  leads to a contradiction.]

(2) (i) Since  $s \succ t$ , it follows that  $s \succeq t$ .  
But  $r \succeq s$  and  $\succeq$  is transitive. Therefore  $r \succeq t$ .

(3) (ii) Assume that  $t \succeq r$ .  
Because  $r \succeq s$  and  $\succeq$  is transitive, it follows that  $t \succeq s$ .  
But we are given that  $s \succ t$ , which implies that  $\neg(t \succeq s)$ .  
This contradiction shows that  $\neg(t \succeq r)$ .  
Since  $r \succeq t$  and  $\neg(t \succeq r)$ , we have that  $r \succ t$ .

- [8 marks] 4. Your utility for an increase in wealth of  $\$x$ , where  $x \geq 0$ , is given by

$$u(x) = \sqrt{x+25} - 5$$

Consider the lottery  $L_1 = \langle \$0, \frac{1}{2}; \$75, \frac{1}{2} \rangle$ , in which you win  $\$75$  with probability  $\frac{1}{2}$  and  $\$0$  with probability  $\frac{1}{2}$ . Answer the following questions, correct to the nearest cent if appropriate.

- Find  $EL_1$ , the expected value of  $L_1$ .
- Find  $Eu(L_1)$ , the expected utility of  $L_1$ .
- Find  $CE(L_1)$ , the certainty equivalent of  $L_1$ .
- Using only your answers to (a), (b), and (c), determine whether you are risk-averse, risk-acceptant, or risk-neutral with respect to  $L_1$ .
- Let  $L_2 = \langle \$0, \frac{1}{4}; \$25, \frac{1}{2}; \$200, \frac{1}{4} \rangle$ . Calculate  $Eu(L_2)$ , and use your answer to determine whether you prefer  $L_1$  or  $L_2$  or are indifferent between them.

(1) (a)  $EL_1 = \frac{1}{2}[0] + \frac{1}{2}[75] = 37.50$

(2) (b)  $Eu(L_1) = \frac{1}{2}u(0) + \frac{1}{2}u(75) = \frac{1}{2} \times 0 + \frac{1}{2} \times 5 = 2.5$

(3) (c)  $CE(L_1) = \text{solution of } u(x) = 2.5$   
 $\sqrt{x+25} - 5 = 2.5 \Rightarrow \sqrt{x+25} = 7.5$   
 $x + 25 = 56.25 \Rightarrow x = 31.25$   
 $\therefore CE(L_1) = \$31.25$

(1) (d) Because  $CE(L_1) < EL_1$ , I am risk-averse

(2) (e)  $Eu(L_2) = \frac{1}{4}u(0) + \frac{1}{2}u(25) + \frac{1}{4}u(200)$   
 $= \frac{1}{4} \cdot 0 + \frac{1}{2}(\sqrt{50} - 5) + \frac{1}{4}(\sqrt{225} - 5) = 3.5355$   
2.0710                      1.0

Because  $Eu(L_2) > Eu(L_1)$ , I prefer  $L_2$  to  $L_1$   
 $L_2 \succ L_1$

- [8 marks] 5. If possible, solve the bimatrix game  $G_5$  by iterated elimination of dominated strategies. When a strategy is dominated, indicate *all* strategies that dominate it, and whether the dominance is strong or weak. What kind of dominance-solvability, if any, applies to  $G_5$ ?

$G_5$

	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$
$r_1$	0, -1	0, 0	-2, 0	2, 0	0, -1
$r_2$	4, 3	2, 3	-1, -2	-1, 1	0, 1
$r_3$	1, 3	2, 4	-1, -2	-1, 1	0, 4
$r_4$	2, 2	3, 2	0, 3	1, 3	1, 2

Step 1

	$c_2$	$c_4$	$c_5$
$r_1$	0, 0	2, 0	0, -1
$r_2$	2, 3	-1, 1	0, 1
$r_4$	3, 2	1, 3	1, 2

- (3) { (1) Eliminate  $r_3$  (weakly dominated by  $r_2$  &  $r_4$ )  
 Eliminate  $c_1$  (weakly dominated by  $c_2$ )  
 Eliminate  $c_3$  (weakly dominated by  $c_4$ )
- (2) { (2) Eliminate  $r_2$  (strictly dominated by  $r_4$ )  
 Eliminate  $c_5$  (weakly dominated by  $c_2$  and  $c_4$ )
- (1) (3) Eliminate  $c_2$  (weakly dominated by  $c_4$ )
- (1) (4) Eliminate  $r_4$  (str. dom. by  $r_1$ )
- (1)  $G_5$  is dominance-solvable (not strict)  
 to  $(r_1, c_4)$  w.i.

Step 2

	$c_2$	$c_4$
$r_1$	0, 0	2, 0
$r_4$	3, 2	1, 3

Step 3

	$c_4$
$r_1$	2, 0
$r_4$	1, 3

- [4 marks] 6. Find all pure-strategy Nash Equilibria of the bimatrix game  $G_5$ . [For convenience,  $G_5$  is reproduced below.]

$G_5$

	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$
$r_1$	0, -1	0, 0	-2, 0	2, 0	0, -1
$r_2$	4, 3	2, 3	-1, -2	-1, 1	0, 1
$r_3$	1, 3	2, 4	-1, -2	-1, 1	0, 4
$r_4$	2, 2	3, 2	0, 3	1, 3	1, 2

$G_5$  has three Nash eq.  
 $(r_2, c_1), (r_1, c_4) \& (r_4, c_3)$

(Must demonstrate method)

-1 each error

- [3 marks] 7. Suppose that II plays mixed strategy  $y = (0, 0, \frac{1}{2}, 0, \frac{1}{2})$ . By determining I's expected utility for each of her pure strategies, determine I's best response to  $y$ ?

(3)

$$Eu_1(r_1, y) = 0 + 0 + \frac{1}{2}[-2] + 0 + \frac{1}{2}[0] = -1$$

$$Eu_1(r_2, y) = 0 + 0 + \frac{1}{2}[-1] + 0 + \frac{1}{2}[0] = -\frac{1}{2}$$

$$Eu_1(r_3, y) = 0 + 0 + \frac{1}{2}[-1] + 0 + \frac{1}{2}[0] = -\frac{1}{2}$$

$$Eu_1(r_4, y) = 0 + 0 + \frac{1}{2}[0] + 0 + \frac{1}{2}[1] = \frac{1}{2}$$

$r_4$  is I's best response to  $y$

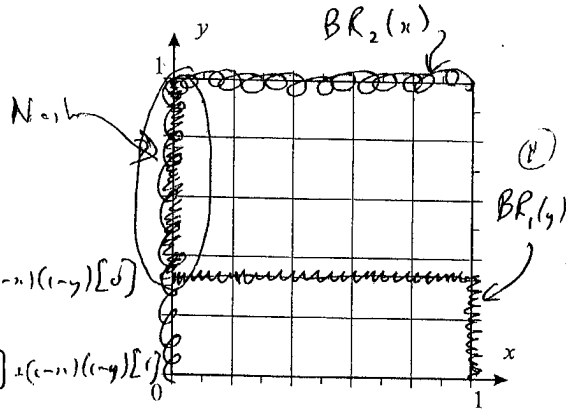
- [12 marks] 8. (a) In the game  $G_8$ , assume that player I's mixed strategy is  $(x, 1-x)$  and that player II's mixed strategy is  $(y, 1-y)$ , as shown adjacent to the game matrix. Show that I's expected utility  $Eu_1$  and II's expected utility  $Eu_2$  are given by

$$\begin{aligned} Eu_1(x, y) &= x + y - 3xy \\ Eu_2(x, y) &= 1 - x + xy \end{aligned}$$

- (b) On the axes provided, sketch the players' best response correspondences. Label each correspondence clearly.  
 (c) Using the best responses, determine all Nash Equilibria of  $G_8$ .  
 (d) For each Nash equilibrium, indicate whether the Nash equilibrium is a pure-strategy or mixed-strategy equilibrium, and calculate the players' expected utilities.

$G_8$

	$c_1$	$c_2$	
$r_1$	-1, 1	1, 0	$(x)$
$r_2$	1, 1	0, 1	$(1-x)$
	$(y)$	$(1-y)$	



(2) (a) 
$$\begin{aligned} \bar{Eu}_1(x, y) &= xy[-1] + x(1-y)[1] + (1-x)y[1] + (1-x)(1-y)[0] \\ &= x + y - 3xy \\ \bar{Eu}_2(x, y) &= xy[1] + x(1-y)[0] + (1-x)y[1] + (1-x)(1-y)[1] \\ &= 1 - x + xy \end{aligned}$$

(5) (b) 
$$\frac{\partial Eu_1}{\partial x} = 1 - 3y \Rightarrow BR_1(y) = \begin{cases} 0 & \text{if } y < \frac{1}{3} \\ \frac{1}{3} & \text{if } y = \frac{1}{3} \\ 1 & \text{if } y > \frac{1}{3} \end{cases}$$

$$\frac{\partial Eu_2}{\partial y} = x \Rightarrow BR_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(2) (c) Pure-strategy Nash Equilibrium  $x=0, y=1 \rightarrow (r_2, c_1) [1, 1]$

(3) (d) Mixed-strategy Nash Eq:  $x=0, \frac{1}{3} \leq y < 1 \rightarrow (0, y)$  when  $\frac{1}{3} \leq y < 1$   

$$Eu_1(0, y) = y \quad Eu_2(0, y) = 1$$

$$[Eu_1, Eu_2] = [y, 1]$$

- [4 marks] 9. In a two-person strategic-form game, denote player I's strategy set by  $S$  and player II's set by  $T$ . Dominance for player I is a relation on  $S$ . Recall that  $s_1 \in S$  dominates  $s_2 \in S$  if and only if

- (1)  $u(s_1, t) \geq u(s_2, t)$  for all  $t \in T$ , and
- (2) there exists  $t^+ \in T$  such that  $u(s_1, t^+) > u(s_2, t^+)$ .

Show that dominance is an asymmetric relation.

Assume that  $s_1$  dominates  $s_2$

- (1) To obtain a contradiction, suppose that  $s_2$  dominates  $s_1$   
 Then  $u_i(s_2, t) \geq u_i(s_1, t)$  for all  $t \in T$   
 and  $u_i(s_2, t^0) > u_i(s_1, t^0)$  for some  $t^0 \in T$  ← contradiction  
 But we have that  $u_i(s_1, t) \geq u_i(s_2, t)$  for all  $t \in T$  ← contradiction  
 (1) Therefore  $s_2$  cannot dominate  $s_1$ , so dominance is asymmetric

[11 marks] 10. For each of the two-person zerosum games  $G_{10a}$  and  $G_{10b}$ ,

- (a) Identify all saddle points, if any;
- (b) Find all optimal strategies for both players, and the value of the game.

$G_{10a}$

2	0	1	-2	-2
-3	-1	0	0	-1
1	0	0	0	0

$G_{10b}$

	$c_1$	$c_2$	$c_3$	$c_4$	
$r$	2	0	3	-2	$x$
$1-x$	-1	2	-1	3	$1-x$

(3)

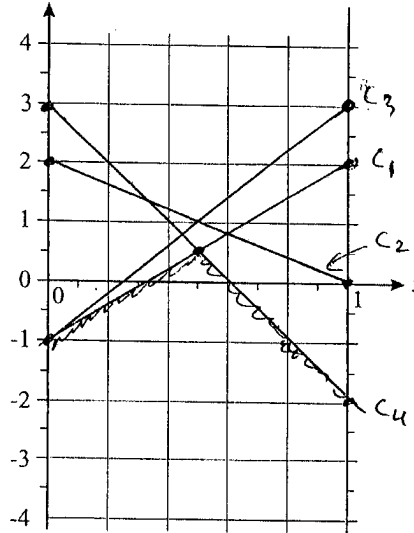
2 0 1 0

Saddle point at  $(r_3, c_2)$  &  $(r_3, c_4)$

value 0

Optimal strategies  $r_3$  for I  
 $c_2$  or  $c_4$  for II

$Eu(x,c)$



(8)

$$Eu(x, c_1) = x[2] + (1-x)[-1] = -1 + 3x$$

$$Eu(x, c_2) = x[0] + (1-x)[2] = 2 - 2x$$

$$Eu(x, c_3) = x[3] + (1-x)[-1] = -1 + 4x$$

$$Eu(x, c_4) = x[-2] + (1-x)[3] = 3 - 5x$$

Maximum of Lower Envelope occurs where  $c_1$  and  $c_4$  intersect

$$-1 + 3x = 3 - 5x \Rightarrow 8x = 4 \Rightarrow x = \frac{1}{2}$$

$$v = \text{height} = -1 + 3 \times \frac{1}{2} = 3 - 5 \times \frac{1}{2} = \frac{1}{2}$$

Best response of II of form  $y^* = (y, 0, 0, 1-y)$

$$Eu(r_1, y^*) = y[2] + 0 + 0 + (1-y)[-2] = \frac{1}{2}$$

$$\Rightarrow 2y - 2 + 2y = \frac{1}{2}$$

$$4y = \frac{5}{2} \Rightarrow y = \frac{5}{8}$$

∴ Optimal for II =  $(\frac{5}{8}, 0, 0, \frac{3}{8})$

[10 marks] 11. Let  $W$  denote the set of all possible outcomes in a two-person game. If  $w \in W$ , let  $u_1(w)$  denote player I's utility for  $w$  and let  $u_2(w)$  denote player II's utility for  $w$ . Recall that  $w_1 \in W$  is Pareto-superior to  $w_2 \in W$  if and only if  $u_i(w_1) \geq u_i(w_2)$  for  $i = 1$  and  $2$ , and  $u_i(w_1) > u_i(w_2)$  for  $i = 1$  or  $2$ .

(a) Prove that Pareto-superiority is a transitive relation.

Suppose  $w_1$  is P-S to  $w_2$  and  $w_2$  is P-S to  $w_3$

We'll show that  $w_1$  is P-S to  $w_3$

(1) For  $i = 1$  and  $2$ ,  $u_i(w_1) \geq u_i(w_2) \geq u_i(w_3) \Rightarrow u_i(w_1) \geq u_i(w_3)$

(2) Suppose that  $u_i(w_1) > u_i(w_2)$ . Since  $u_i(w_2) \geq u_i(w_3)$

we have  $u_i(w_1) > u_i(w_2) \geq u_i(w_3) \Rightarrow u_i(w_1) > u_i(w_3)$

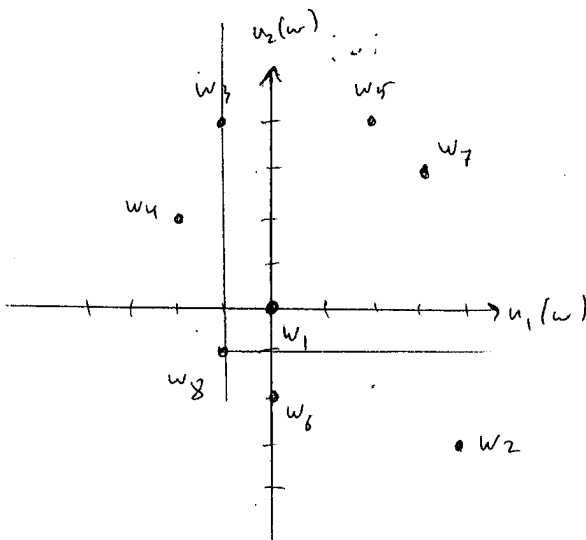
(b) In  $G_{11}$ , the set of outcomes is  $W_{11} = \{w_1, w_2, \dots, w_8\}$ . The players' utilities are as given in the table below. For example, at  $w_2$ , the utility profile is  $[4, -3]$ .

$w$	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$	$w_7$	$w_8$
$u_1(w)$	0	4	-1	-2	2	0	3	-1
$u_2(w)$	0	-3	4	2	4	-2	3	-1

(i) Find all outcomes that are Pareto-superior to  $w_8$ .

(ii) Find all Pareto-optimal outcomes.

(iii) Prove that Pareto-superiority is not weakly complete by finding a counterexample from  $W_{11}$ .



(a) It is clear that  $w_1, w_3, w_5$ , and  $w_7$  are Pareto-superior to  $w_8$

(b) Pareto-optimal outcomes are  $w_2, w_5$ , and  $w_7$

(c)  $w_5$  is not Pareto-superior to  $w_7$  and  $w_7$  is not Pareto-superior to  $w_5$   
 $\therefore$  Pareto-superiority is not weakly complete