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Fall Term, 2009

Name: Model Answers

Student Number: ____________

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 235 – Introduction to Game Theory

Test I – October 13, 2009

Instructor: *Dr. M. Kilgour*

Time Allowed: *80 minutes*

Total Value: *80 marks*

Number of Pages: *6 plus cover page*

Instructions:

Non-programmable, non-graphing calculators are permitted. No other aids are allowed.

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. ***Please note that questions are printed on both sides of the test pages.***

Show all your work. Insufficient justification will result in a loss of marks.

[*8 marks*] 1. Let $W = \{1, 2, 3, 4\}$. The relation B is defined on W by $x\ B\ y$ if and only if the accompanying table contains an X in the cell (x, y). Answer the following questions about the relation B. Justify your answers.

	$y = 1$	$y = 2$	$y = 3$	$y = 4$
$x = 1$	X	X		X
$x = 2$			X	
$x = 3$		X		X
$x = 4$	X			X

(a) Is B reflexive?

No. Because $\neg 2 B 2$
(or $\neg 3 B 3$)

2

(b) Is B symmetric?

No Because $1 B 2$ but $\neg 2 B 1$
(or $3 B 4$ but $\neg 4 B 3$)

2

(c) Is B antisymmetric?

No. Because $2 B 3$ and $3 B 2$ (but $2 \neq 3$).
(or $4 B 1$ and $1 B 4$ (but $4 \neq 1$).)

2

(d) Is B transitive?

No. Because $1 B 2$ and $2 B 3$, but $\neg 1 B 3$
(or $3 B 4$ and $4 B 1$, but $\neg 4 B 1$)
3

2

[*5 marks*] 2. (a) Let B be any relation on a set W. Prove that if B is complete, then B is also reflective.

For any x & y either $x B y$ or $y B x$
(Must substitute $y = x$ in defn)
Suppose $y = x$. Then either ($x B x$ or $x B x$) $\Rightarrow$ $x B x$
Therefore, for any $x \in W$, $x B x$, so B is reflexive

3

(b) Consider the relation of B of #1. Can you use 2(a) and your answers to #1 to determine whether B is complete?

Yes, If B were complete, B would be reflexive. (by 2(a))
But B is not reflexive (by 1(a)). Must use 2(a) & 1(a)
Therefore B cannot be complete.

2

If B is a relation on W, then

B is *reflexive* if $x\ B\ x$ for all $x \in W$.

B is *irreflexive* if $x \neg B\ x$ for all $x \in W$.

B is *symmetric* if $x\ B\ y$ implies $y\ B\ x$ for all $x \in W$ and $y \in W$.

B is *asymmetric* if $x\ B\ y$ implies $y \neg B\ x$ for all $x \in W$and $y \in W$.

B is *antisymmetric* if $x\ B\ y$ and $y\ B\ x$ implies $x = y$ for all $x \in W$ and $y \in W$.

B is *transitive* if $x\ B\ y$ and $y\ B\ z$ implies $x\ B\ z$ for all $x \in W$, $y \in W$, and $z \in W$.

B is *complete* if for every $x \in W$ and $y \in W$, either $x\ B\ y$ or $y\ B\ x$.

B is *weakly complete* if for every $x \in W$ and $y \in W$such that $x \neq y$, either $x\ B\ y$ or $y\ B\ x$.

Over

[*6 marks*] 3. Assume a weak preference relation $\succeq$ on a set W, i.e., assume that $\succeq$ is reflexive, transitive, and complete. Define a strict preference relation $\succ$ on W by

$$x \succ y \text{ if and only if } x \succeq y \text{ and } \neg(y \succeq x)$$

for any $x, y \in W$. Assume that $r, s, t \in W$ and that $r \succeq s$ and $s \succ t$. Prove that $r \succ t$.

[**Hint:** Carry out the following steps: (i) Prove that $r \succeq t$. (ii) Show that the assumption that $t \succeq r$ leads to a contradiction.]

2 (i) Because $s \succ t$, it must be the case that $s \succsim t$

Because $r \succsim s$ and $\succsim$ is transitive, $r \succsim t$.

(ii) Suppose that $t \succsim r$.

Because $r \succsim s$ and $\succsim$ is transitive, $t \succsim s$.

4 But, by assumption $s \succ t$, so $s \succsim t$ and $\neg(t \succsim s)$

This contradiction proves that $\neg(t \succsim r)$.

$\therefore$ $r \succsim t$ and $\neg(t \succsim r)$, so $r \succ t$.

[*8 marks*] 4. Your utility for an increase in wealth of \$$x$, where $0 \leq x \leq 100$, is given by

$$u(x) = \sqrt{x+9} - 3$$

Consider the lottery $L_1 =<$ \$0, ½; \$40, ½ $>$, in which you win \$40 with probability ½ and \$0 with probability ½.

(a) Find EL_1, the expected value of L_1.

(b) Find $Eu[L_1]$, the expected utility of L_1.

(c) Find $CE(L_1)$, the certainty equivalent of L_1. Your answer should be correct to the nearest cent.

(d) Using only your answers to (a), (b), and (c), determine whether you are risk-averse, risk-acceptant, or risk-neutral with respect to L_1.

(e) Let $L_2 =<$ \$0, ¼; \$7, ½; \$72, ¼ $>$. Calculate $Eu(L_2)$, and use your answer to determine whether you prefer L_1 or L_2, or are indifferent between them.

1 (a) $EL_1 = \frac{1}{2}[\$0] + \frac{1}{2}[\$40] = \$20.$

2 (b) $Eu[L_1] = \frac{1}{2}u(0) + \frac{1}{2}u(40) = \frac{1}{2}[0] + \frac{1}{2}[4] = 2$

3 (c) $CE(L_1) =$ solution of $u(x) = 2$ 1

$\sqrt{x+9} - 3 = 2 \Rightarrow \sqrt{x+9} = 5$

$x + 9 = 25 \Rightarrow x = 16$

$\therefore CE(L_1) = \$16$

1 (d) Since $CE(L_1) < EL_1$, I am risk-averse ← justification req'd

(e) $Eu(L_2) = \frac{1}{4}u(0) + \frac{1}{2}u(7) + \frac{1}{4}u(72) = \frac{1}{4}[0] + \frac{1}{2}[1] + \frac{1}{4}[6] = 2$

1

Since $Eu(L_1) = Eu(L_2)$, I am indifferent between L_1 and L_2.

justification req'd

$1.5 \times 2 \times 3$

[*8 marks*] 5. If possible, solve the bimatrix game G_5 by iterated elimination of dominated strategies. When a strategy is dominated, indicate *all* strategies that dominate it, *and* whether the dominance is strict or weak. [**Hint:** Be certain to find all dominated strategies before reducing the bimatrix.] What kind of dominance-solvability, if any, applies to G_5.

G_5

	c_1	c_2	c_3	c_4
r_1	$2,0$	$0,0$	$0,0$	$0,-1$
r_2	$1,1$	$3,0$	$0,1$	$3,0$
r_3	$-1,1$	$3,4$	$-1,-2$	$3,3$
r_4	$-1,1$	$2,3$	$-1,-2$	$6,3$

After Step 1

	c_1	c_2
r_1	2, 0	0, 0
r_2	1, 1	3, 0
r_4	−1, 1	2, 3

3 (1) Eliminate r_3 (weakly dominated by r_2)
Eliminate c_3 (weakly dominated by c_1)
Eliminate c_4 (weakly dominated by c_2)

1 (2) Eliminate r_4 (strictly dominated by r_2)

1 (3) Eliminate c_2 (weakly dominated by c_1)

1 (4) Eliminate r_2 (strictly dominated by r_1)

After Step 2

	c_1	c_2
r_1	2, 0	0, 0
r_2	1, 1	3, 0

After Step 3

	c_1
r_1	2, 0
r_2	1, 1

2 ∴ G_5 is dominance-solvable (not strict dominance-solvable) to (r_1, c_1), with utility profile $[2, 0]$ ← not req'd

[*4 marks*] 6. Find all pure-strategy Nash Equilibria of the bimatrix game G_5. [For convenience, G_5 is reproduced below.]

G_5

	c_1	c_2	c_3	c_4
r_1	$\underline{2},\underline{0}$	$0,\underline{0}$	$\underline{0},\underline{0}$	~~$0,-1$~~
r_2	$1,\underline{1}$	$\underline{3},0$	$\underline{0},\underline{1}$	~~$3,0$~~
r_3	~~$-1,1$~~	$\underline{3},\underline{4}$	~~$-1,-2$~~	~~$3,3$~~
r_4	~~$-1,1$~~	$2,\underline{3}$	~~$-1,-2$~~	$\underline{6},\underline{3}$

4

Must demonstrate method
−1 each error

G_5 has five (5) pure-strategy Nash Equilibria (r_1, c_1), (r_1, c_3), (r_2, c_3), (r_3, c_2), and (r_4, c_4)

[*4 marks*] 7. If player I plays mixed strategy $x = (0, {}^1\!/\!_2, {}^1\!/\!_2, 0)$, determine II's expected utility for each of II's pure strategies. Which pure strategy is II's best response to x?

4

$Eu_2(x, c_1) = 0 + \frac{1}{2}[1] + \frac{1}{2}[1] + 0 = 1$

$Eu_2(x, c_2) = 0 + \frac{1}{2}[0] + \frac{1}{2}[4] + 0 = 2$

$Eu_2(x, c_3) = 0 + \frac{1}{2}[1] + \frac{1}{2}[-2] + 0 = -\frac{1}{2}$

$Eu_2(x, c_4) = 0 + \frac{1}{2}[0] + \frac{1}{2}[3] + 0 = \frac{3}{2}$

3

1 Answer: c_2 is II's best pure-strategy response to x.

2:30 − 5:30

Over

[*12 marks*] 8. (a) In the game G_8, assume that player I's mixed strategy is $(x, 1-x)$ and that player II's mixed strategy is $(y, 1-y)$, as shown in the game matrix. Show that I's expected utility Eu_1 and II's expected utility Eu_2 are given by

$$\begin{aligned} Eu_1(x,y) &= 2 - 3x - 2y + 5xy \\ Eu_2(x,y) &= x + y - xy \end{aligned}$$

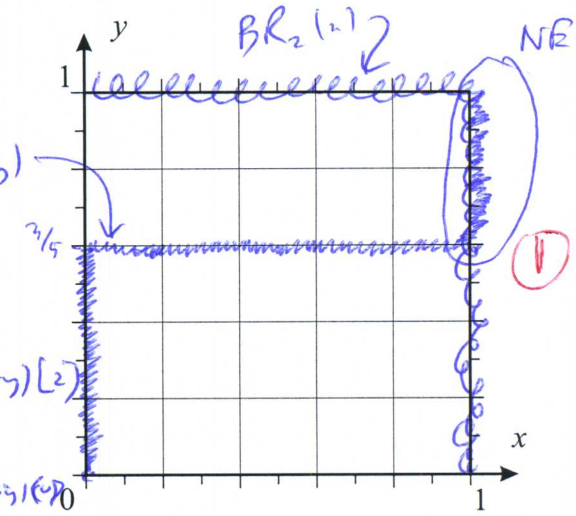
(b) On the axes provided, sketch the players' best response correspondences. Label each correspondence clearly.

(c) Using the best responses, determine all Nash Equilibria of G_8.

(d) Indicate whether each Nash equilibrium is a pure-strategy or a mixed-strategy equilibrium, and calculate the players' expected utilities.

G_8

	c_1	c_2	
r_1	2, 1	−1, 1	(x)
r_2	0, 1	2, 0	$(1-x)$
	(y)	$(1-y)$	



(2) (a) $Eu_1(x,y) = xy[2] + x(1-y)[-1] + (1-x)y[0] + (1-x)(1-y)[2]$

(1) $= 2 - 3x - 2y + 5xy$

(1) $Eu_2(x,y) = xy[1] + x(1-y)[1] + (1-x)(y)[1] + (1-x)(1-y)[0]$

$= x + y - xy$

(5) (b) $\dfrac{\partial Eu_1}{\partial x} = -3 + 5y$ so $BR_1(y) = \begin{cases} 1 & \text{if } y > 3/5 \\ [0,1] & \text{if } y = 3/5 \\ 0 & \text{if } y < 3/5 \end{cases}$ (2)

$\dfrac{\partial Eu_2}{\partial y} = 1 - x$ so $BR_2(x) = \begin{cases} 1 & \text{if } x < 1 \\ [0,1] & \text{if } x = 1 \end{cases}$ (2)

(5) (c) (d) Pure-strategy Nash Equilibrium $(x,y) = (1,1)$, (r_1, c_1), utility $[2, 1]$ ← (2)

Mixed-strategy Nash Equilibria $(x,y) = (1,y)$ for $3/5 \le y < 1$ (3)

$Eu_1(1,y) = 2 - 3 - 2y + 5y = -1 + 3y$ Exp. Utility profile $[-1+3y, 1]$

$Eu_2(1,y) = 1 + y - y = 1$

[*4 marks*] 9. In a two-person strategic-form game, player I's strategy set is S and player II's is T. Recall that dominance for I is a relation on S, and that $s_1 \in S$ dominates $s_2 \in S$ if and only if

(1) $u(s_1, t) \ge u(s_2, t)$ for all $t \in T$, and

(2) there exists $t^+ \in T$ such that $u(s_1, t^+) > u(s_2, t^+)$.

Show that dominance is an asymmetric relation.

Suppose s_1 dominates s_2. Then $u_1(s_1, t) \ge u_1(s_2, t)$ for all $t \in T$ and $u_1(s_1, t^+) > u_1(s_2, t^+)$ for some $t \in T$

(2) for looking for contradiction

Now suppose s_2 also dominates s_1. Then $u_1(s_2, t) \ge u_1(s_1, t)$ for all $t \in T$ and $u_1(s_2, t^0) > u_1(s_1, t^0)$ for some $t^0 \in T$

(4)

(2) for demonstrating contradiction

But $u_1(s_1, t^0) < u_1(s_2, t^0)$ for some $t^0 \in T$

contradicts $u_1(s_1, t) \ge u_1(s_2, t)$ for all $t \in T$

This contradiction shows that s_2 cannot dominate s_1

So dominance is asymmetric.

$6\frac{1}{2}$

8:30

[*11 marks*] 10. For each of the two-person zerosum games G_{10a} and G_{10b},

(a) Identify all saddle points, if any;

(b) Find all optimal strategies for both players, and the value of the game.

G_{10a} (3)

2	0	−1	−1
−3	1	0	−3
1	0	0	0
2	1	0	

G_{10b} (8)

	c_1	c_2	c_3	c_4		
	3	2	0	−2	−2	$p(x)$
	−1	−1	3	4	−1	$(1-x)$
	3	2	3	4		

(1) G_{10a} : Unique saddle point (r_3, c_3) (1)
Optimal strategy for I : r_3
(1) Optimal strategy for II : c_3
Value = 0

must specify

G_{10b} : No saddle points (1)

$Eu_1(x, c_1) = 3x + (-1)(1-x) = -1 + 4x$
$Eu_1(x, c_2) = 2x + (-1)(1-x) = -1 + 3x$
$Eu_1(x, c_3) = 0x + 3(1-x) = 3 - 3x$
$Eu_1(x, c_4) = -2 + 4(1-x) = 4 - 6x$ (2)

See diagram below

Maximum of Lower Envelope (1)
where c_2 & c_4 intersect

$-1 + 3x = 4 - 6x \Rightarrow x = 5/9$ (1) ← Optimal Strategy for I $= (5/9, 4/9)$

Value $-1 + 3(5/9) = 6/9 = 2/3$ (1)

Optimal Strategy for II must have the form $(0, y, 0, 1-y)$ (1)

For r_1, Solving $0[3] + y[2] + 0[0] + (1-y)[-2] = 2/3$

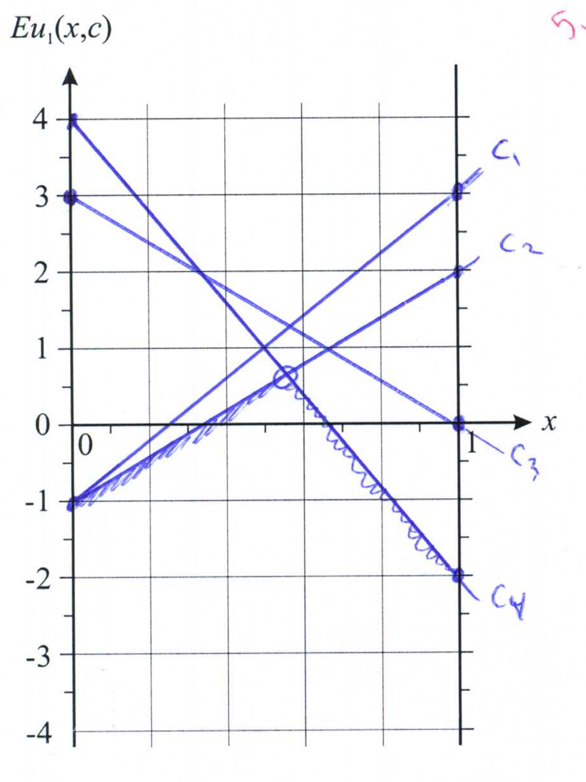
$-2 + 4y = 2/3$

$4y = 8/3$

$y = 2/3$

$5y - \frac{10}{3}$ $-5y+4$ $2/3$

$\therefore$ Optimal Strategy of II
$= (0, 2/3, 0, 1/3)$ (1)



$3\frac{1}{2} + 6\frac{1}{2} + \frac{1}{2}$

$-y + 4(1-y)$
$-5y + 4 = 2/3$
$-5y = 2/3 - 4$
$= 2/3 - 12/3$
$-5y = -10/3$

12 8 $\frac{15}{9} - \frac{9}{9}$

$10\frac{1}{2}$ + 9:10 − 11:10 9:10 7:45 − 8

Over

[*10 marks*] 11. Let W denote the set of all possible outcomes in a game with two players, called I and II. If $w \in W$, denote player I's utility for w by $u_1(w)$ and player II's utility for w by $u_2(w)$. Recall that $w_1 \in W$ is *Pareto-superior* to $w_2 \in W$ if and only if $u_i(w_1) \geq u_i(w_2)$ for $i = 1$ and 2, and $u_i(w_1) > u_i(w_2)$ for $i = 1$ or 2. Also, $w_1 \in W$ is *Pareto-optimal* if and only if there is no outcome $w \in W$ such that w is Pareto-superior to w_1.

(a) Prove that Pareto-superiority is an transitive relation.

Suppose w_1 is Pareto-superior to w_2 & w_2 is Pareto-superior to w_3

4

2 (1) $u_i(w_1) \geq u_i(w_2)$ for $i=1$ & 2 & $u_i(w_2) \geq u_i(w_3)$ for $i=1$ & 2

$\therefore$ $u_i(w_1) \geq u_i(w_3)$ for $i=1$ and 2.

2 (2) For some player i^*, $u_{i^*}(w_1) > u_{i^*}(w_2)$

Since $u_{i^*}(w_2) \geq u_{i^*}(w_3)$, it follows that $u_{i^*}(w_1) > u_{i^*}(w_3)$

$\therefore$ w_1 is Pareto superior to w_3, so Pareto-superiority is transitive.

(b) Suppose that $W = \{w_1, w_2, \ldots, w_8\}$, and that the players' utilities are as given in the table below. For example, at w_2, the utility profile is $[4, -3]$.

w	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
$u_1(w)$	0	4	-2	2	-1	2	2	3
$u_2(w)$	0	-3	1	-2	-2	0	1	-1

(i) Find all outcomes that are Pareto-superior to w_5.

(ii) Find all Pareto-optimal outcomes.

(iii) Prove that Pareto-superiority is not weakly complete by finding a counterexample from W.

$\forall x, y \in W$, $x \neq y$: xBy or yBx

As shown in the sketch

6

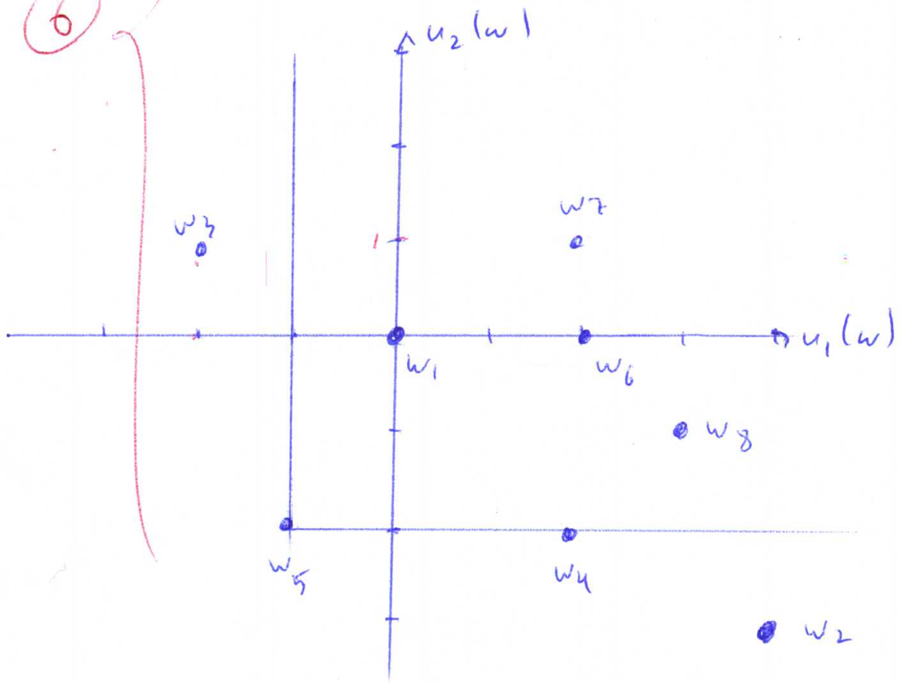
(i) w_1, w_4, w_6, w_7, & w_8 are all Pareto-superior to w_5 2

(ii) The Pareto-optimal outcomes are w_7, w_8, and w_2. 2

(iii) Note that w_1 is not Pareto-superior to w_2 and w_2 is not Pareto-superior to w_1 2

$\therefore$ Pareto-superiority is not weakly complete

(Many other examples)



Sketch not required

8/11:10