

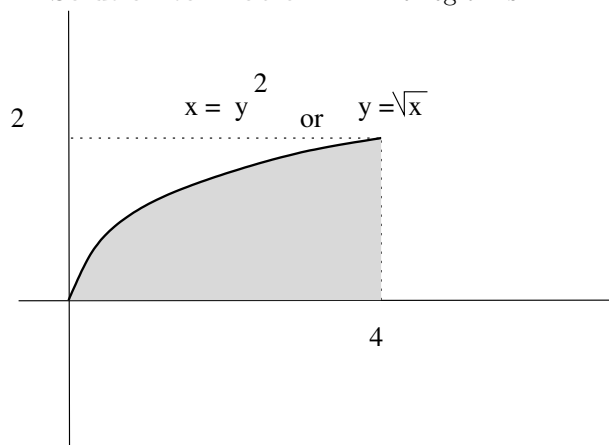
Midterm Exam Emat 233, March 18 2005

Special instructions: solve five problems of your choice

[10 points] **Problem 1.** Evaluate the following integral by reversing the order of integration

$$\int_0^2 \int_{y^2}^4 \cos(\sqrt{x^3}) dx dy$$

Solution to Problem 1. The region is

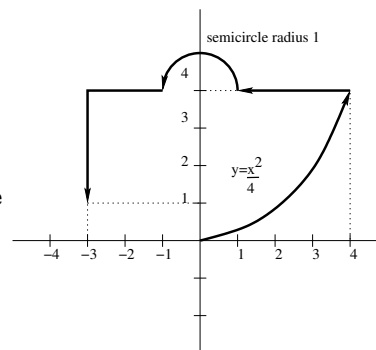


$$\int_0^4 \int_0^{\sqrt{x}} \cos(\sqrt{x^3}) dy dx = \int_0^4 \sqrt{x} \cos(\sqrt{x^3}) dx = \frac{2}{3} \int_0^4 \cos(x^{\frac{3}{2}}) dx^{\frac{3}{2}} = \frac{2}{3} \sin(x^{\frac{3}{2}}) \Big|_0^4 = \frac{2}{3} \sin(8)$$

[10 points] **Problem 2.** Consider the vector field in 2-dimensions

$$\vec{F}(x, y) = (xe^{-x} + 2xy) \mathbf{i} + (y + x^2) \mathbf{j}.$$

Compute the line integral of $\vec{F}$ along the curve $\mathcal{C}$ in the picture. **Motivate** your answer if necessary.



Solution to Problem 2. The contour is rather complicated so I would suspect a shortcut to exist. Indeed we check that the integral is independent of the path

$$\begin{aligned} \partial_y (xe^{-x} + 2xy) &= 2x \\ \partial_x (y + x^2) &= 2x \quad OK. \end{aligned}$$

So we look for the potential ϕ .

$$\phi = \int (xe^{-x} + 2xy) dx = -xe^{-x} + \int e^{-x} dx + x^2 y + h(y) = -xe^{-x} - e^{-x} + x^2 y + h(y)$$

Then

$$\partial_y \phi = x^2 + h'(y) = x^2 + y \Rightarrow h'(y) = y \Rightarrow h = \frac{y^2}{2} + C$$

Thus a suitable potential is

$$\phi(x, y) = -xe^{-x} - e^{-x} + x^2y + \frac{y^2}{2}$$

The integral then evaluates to

$$\phi(x, y) \Big|_{(0,0)}^{(-3,1)} = \phi(-3, 1) - \phi(0, 0) = 2e^3 + 9 + \frac{1}{2} + 1 = 2e^3 + \frac{21}{2}.$$

[10 points] Problem 3. By using the appropriate theorem (which must be named) compute the circulation $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ of the vector field

$$\vec{F}(x, y) = (3y - e^{-x^2})\mathbf{i} + (5x + \ln(3 + \cos^2(y^{45})))\mathbf{j}$$

around the curve $\mathcal{C}$ given by the circle of radius 2 and center $(3, 2)$ oriented counterclockwise.

Solution to Problem 3. The theorem is Green's theorem. So we have

$$\begin{aligned} & \oint_{\mathcal{C}} (3y - e^{-x^2}) dx + (5x + \ln(3 + \cos^2(y^{45}))) dy = \\ &= \iint_{\mathcal{R}} [\partial_x (5x + \ln(3 + \cos^2(y^{45}))) - \partial_y (3y - e^{-x^2})] dA = \iint_{\mathcal{R}} (5 - 3) dA = 2 \iint_{\mathcal{R}} dA. \end{aligned}$$

The area of the circle is $\pi r^2 = 4\pi$ so we have

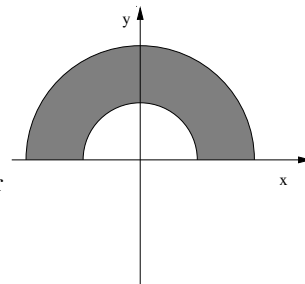
$$2 \iint_{\mathcal{R}} dA = 8\pi.$$

[10 points] Problem 4.

Compute the double integral

$$\iint_{\mathcal{R}} e^{-x^2-y^2} dA$$

over the region consisting of the half washer in the picture with inner radius 1 and outer radius 2.



Solution to Problem 4. The shape of the region suggests passing to polar coordinates, in which the integral reads

$$\int_0^\pi d\theta \int_1^2 e^{-r^2} r dr = \frac{\pi}{2} e^{-r^2} \Big|_1^2 = \frac{\pi}{2} (e^{-4} - e^{-1})$$

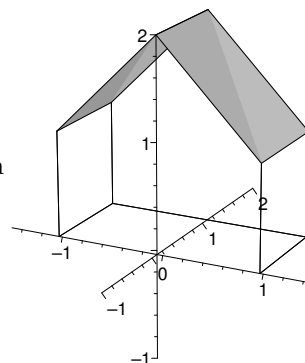
[10 points] Problem 5.

Compute the flux of the vector field

$$\vec{F}(x, y, z) = (3yz)\mathbf{i} + \ln(1 + x^2y^2z^2)\mathbf{j} + z\mathbf{k}$$

across the “roof” surface $\mathcal{S}$ shown in the picture and described hereafter which is the union of portion of two planes, oriented upwards

$$\begin{aligned} z &= 2 + x, & -1 \leq x \leq 0, & \quad 0 \leq y \leq 1 \\ z &= 2 - x, & 0 \leq x \leq 1, & \quad 0 \leq y \leq 1. \end{aligned}$$



Solution to Problem 5. The normal of the first portion of plane is

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$$

and the element of area is

$$dS_1 = \sqrt{1+1}dA = \sqrt{2}dA .$$

Hence we have

$$\begin{aligned} & \iint_{S_1} \frac{(-3yz + z)}{\sqrt{2}} dS_1 = \iint_{\mathcal{R}_1} (2+x)(1-3y) dx dy = \\ & = \int_0^1 \int_{-1}^0 (2+x)(1-3y) dx dy = \int_0^1 (1-3y) \left(2x + \frac{x^2}{2}\right) \Big|_{-1}^0 dy = \int_0^1 (1-3y) \frac{3}{2} dy = \frac{3}{2} \left(y - \frac{3}{2}y^2\right) \Big|_0^1 = -\frac{3}{2} \frac{1}{2} = -\frac{3}{4} \end{aligned}$$

For the second surface we have

$$\mathbf{n}_2 = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$$

and the element of area is

$$dS_2 = \sqrt{1+1}dA = \sqrt{2}dA .$$

Hence we have

$$\begin{aligned} & \iint_{S_2} \frac{(3yz + z)}{\sqrt{2}} dS_2 = \iint_{\mathcal{R}_2} (2-x)(1+3y) dx dy = \\ & = \int_0^1 \int_0^1 (2-x)(1+3y) dx dy = \int_0^1 (2-x) dx \int_0^1 (1+3y) dy = \left(2x - \frac{x^2}{2}\right) \Big|_0^1 \left(y + \frac{3}{2}y^2\right) \Big|_0^1 = \frac{15}{4} \end{aligned}$$

The total flux is thus

$$-\frac{3}{4} + \frac{15}{4} = 3$$

[10 points] Problem 6 Evaluate the following integral by using polar coordinates

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \int_0^{\sqrt{\pi-x^2}} \sin(x^2 + y^2) dy dx$$

Solution to Problem 6. The region is the upper semidisk of radius $\sqrt{\pi}$ and hence we have

$$\int_0^\pi d\theta \int_0^{\sqrt{\pi}} \sin(r^2) r dr = \frac{\pi}{2} \int_0^{\sqrt{\pi}} \sin(r^2) dr^2 = -\frac{\pi}{2} \cos(r^2) \Big|_0^{\sqrt{\pi}} = -\frac{\pi}{2} (\cos(\pi) - \cos(0)) = \pi .$$