MATH1005 — Review

Part 1: Differential Equations

Part 1.1: First-order Differential Equations

1. Separable DE: $\frac{dy}{dx} = f(x)g(y), \Rightarrow dy/g(y) = f(x)dx$.
2. Homogeneous eqn: $\frac{dy}{dx} = g(\frac{y}{x})$. Let $v = \frac{y}{x}$. Then it is separable.
3. 1st order linear eqn: $\frac{dy}{dx} + P(x)y = Q(x)$. Find the integrating factor $I(x) = e^{\int P(x)dx}$. Solution is $y(x)I(x) = \int I(x)Q(x)dx + C$.
4. Bernoulli eqn: $\frac{dy}{dx} + P(x)y = Q(x)y^n$. Let $u = y^{1-n}$, then $\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$, which is a 1st order linear eqn.
5. Exact eqn: $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$ with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. The solution is $f(x,y) = C$, where $f_x = P$, $f_y = Q$. If the equation is not exact, then change the eqn to exact by multiplying integrating factor I:
   - $I = \exp\left\{\int \frac{P_y - Q_x}{Q}\, dx\right\}$, if $\frac{P_y - Q_x}{Q}$ is a function of x only;
   - $I = \exp\left\{-\int \frac{P_y - Q_x}{P}\, dy\right\}$, if $\frac{P_y - Q_x}{P}$ is a function of y only.

Part 1.2: Second-order Differential Equations

1. Homogeneous linear DE with constant coefficients: $ay'' + by' + cy = 0$. Let r_1 and r_2 be two solutions of the Auxiliary eqn $ar^2 + br + c = 0$.
   - $r_1 \neq r_2$: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
   - $r_1 = r_2$: $y = (C_1 + C_2 x)e^{rx}$.
   - $r = \alpha + i\beta$: $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.
2. Reduction of order: If y_1 is a solution of $y'' + p(x)y' + q(x)y = 0$. Then $y_2 = u(x)y_1$, where $u' = \frac{1}{y_1^2} e^{-\int P(x)dx}$.

3. Non-homogeneous DE: $ay'' + by' + cy = G(x)$. Then $y(x) = y_p(x) + y_c(x)$, where $y_p(x)$ is a particular solution of the equation, $y_c(x)$ is the general solution of the complementary equation $ay'' + by' + cy = 0$, To find $y_p(x)$, we use **Variation of parameters**: If $y_c = C_1y_1 + C_2y_2$, then $y_p = u_1y_1 + u_2y_2$, where

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, \\ u_1'y_1' + u_2'y_2' = \frac{G(x)}{a}. \end{cases}$$

4. Cauchy-Euler eqn: $x^2y'' + Axy' + By = 0$. The auxiliary equation is $r^2 + (A-1)r + B = 0$.
   - If $r_1 \neq r_2$ are real, then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$.
   - If $r_1 = r_2$ (real), then $y_1 = x^{r_1}$ and $y_2 = x^{r_1}\ln(x)$.
   - If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $y_1 = x^\alpha \cos[\beta\ln(x)]$ and $y_2 = x^\alpha \sin[\beta\ln(x)]$.

5. Systems of Equations: Let x and y be functions of t. A system of two equations with constant coefficients has the form

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned}$$

Letting $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set $\det(\lambda I - A) = 0$, and solve for λ. For each eigenvalue λ, the corresponding eigenvectors are the nonzero solutions $\mathbf{v}$ of the matrix equation $(\lambda I - A)\mathbf{v} = 0$. If the matrix A has two independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2$ are two independent solutions of the matrix equation, with the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, where c_1 and c_2 are arbitrary constants. If A has only one independent eigenvector $\mathbf{K}$, then one solution is given by $\mathbf{x}_1 = e^{\lambda t}\mathbf{K}$. A second linearly independent solution is given by $\mathbf{x}_2 = te^{\lambda t}K + e^{\lambda t}P$, where P satisfies $(A - \lambda I)P = K$.

Part 2: Series

1. Sequences $a_1, a_2, ..., a_n, ...$, a_n is the nth term. If $\lim_{n\to\infty} a_n$ exists, then we say the sequence converges. Otherwise, we say the sequence diverges.

2. Series:

   Partial sum: $S_n = \sum_{j=1}^n a_j$. Then $\sum_{j=1}^\infty a_j = S \Leftrightarrow \lim_{n\to\infty} S_n = S$.

3. Geometric series: if $|r| < 1$ then $\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$.

4. Partial fraction: $\frac{k}{n(n+k)} = \frac{1}{n} - \frac{1}{n+k}$.

5. Harmonic series $\sum_{j=1}^{\infty} \frac{1}{n}$ is divergent.

6. Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{j=1}^{\infty} a_j$ is divergent.

7. Integral test: If $f(x)$ is continuous, positive, decreasing, $f(j) = a_j$, then

$$\sum_{j=1}^{\infty} a_j \text{ is convergent} \Leftrightarrow \int_1^{\infty} f(x)dx \text{ is convergent.}$$

8. The $p-$series: $\sum_{j=1}^{\infty} \frac{1}{j^p}$ is: convergent if $p > 1$ and divergent if $p \leq 1$.

9. Direct comparison test: If $0 \leq a_j \leq b_j$, then

$$\sum_{j=1}^{\infty} b_j \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

$$\sum_{j=1}^{\infty} a_j \text{ is divergent} \Rightarrow \sum_{j=1}^{\infty} b_j \text{ is divergent.}$$

10. Limit Comparison test: If $a_n > 0$, $b_n > 0$ and

$$\lim \frac{a_n}{b_n} = c > 0,$$

then

$$\sum a_j \text{ is divergent} \Leftrightarrow \sum b_j \text{ is divergent.}$$

11. Ratio test: Consider the series $\sum_{j=1}^{\infty} a_j$, with $\lim_{j\to\infty} |\frac{a_{j+1}}{a_j}| = L$.
    - If $L < 1$, then the series is absolutely convergent;
    - If $L > 1$, then the series is divergent.

12. Root test: Consider the series $\sum_{j=1}^{\infty} a_j$, with $\lim_{j\to\infty} \sqrt[n]{|a_j|} = L$.
    - If $L < 1$, then the series is absolutely convergent;
    - If $L > 1$, then the series is divergent.

13. Alternating series Test: The alternating series $\sum_{j=1}^{\infty}(-1)^{j-1}b_j$ is convergent if

   (a) $b_j > 0$

   (b) b_j decreasing ($b_1 \geq b_2 \geq b_3 \geq \cdots$)

   (c) $\lim_{j\to\infty} b_j = 0$.

14. Remainder estimate: Let $R_n = \sum_{j=n+1}^{\infty} a_j$. Then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

   For the alternating series, we can estimate the **remainder** by using the following inequality

$$|R_n| \leq b_{n+1}.$$

15. Absolute and conditional convergence:

$$\sum_{j=1}^{\infty} |a_j| \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

   If $\sum_{j=1}^{\infty} a_j$ is convergent, but $\sum_{j=1}^{\infty} |a_j|$ is divergent, then $\sum_{j=1}^{\infty} a_j$ is conditionally convergent.

16. Power Series: Consider the series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

   - Radius of convergence: Using Ratio Test to find R.
   - Interval of convergence: $I = \{a - R, a + R\}$, symmetric to the center a, with two end points $a - R$ and $a + R$. The convergence or divergence at the two end points $x = a - R$ and $x = a + R$ should be checked.

17. Taylor polynomial: Taylor polynomial of degree n approximating $f(x)$ for x at a:

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

   If $n = 1$, we have the linear approximation.

18. Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R.$$

19. Mactaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad |x| < R.$$

20. Series for some special functions such as: e^x, $\sin x$, $\cos x$, $\ln(1+x)$.

21. Binomial series: If p is a real number and $|x| < 1$, then

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \cdots = \sum_{n=0}^{\infty} \binom{p}{n} x^n,$$

here

$$\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}, \quad \binom{p}{0} = 1.$$

Application: Let $f(x) = (1+x)^p$, then

$$f^{(n)}(0) = \binom{p}{n} n! = p(p-1)\cdots(p-n+1).$$

22. Represent functions as Taylor series:

(a) By substitution by using known Taylor series. Basic result:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

(b) Term-by-term differentiation:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \Rightarrow f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

and f and f' have the same radius of convergence.

(c) Term-by-term integration:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \Rightarrow \int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1},$$

and f and $\int f dx$ have the same radius of convergence.

23. Error in Taylor polynomial approximation: Let $P_n(x)$ be the Taylor approximation of $f(x)$ at $x = a$, then Taylor's inequality (The Lagrange Error Bound): If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then on $|x - a| \leq d$,

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}.$$

24. Fourier series

- The (full) Fourier series of $2L$-periodic function $f(x)$ can be written as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right\},$$

  where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx.$$

- Given a function $f(x)$ defined on $(0, L)$. The (half-range) Fourier sine series is

$$\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx.$$

- Given a function $f(x)$ defined on $(0, L)$. The (half-range) Fourier cosine series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx.$$