MAT 2322 C
CALCULUS III
MIDTERM
February 27, 2014

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:______________________________

Instructions:
- Print your name and student number on this page.
- Verify that your copy of the exam has all 6 pages.
- There are 5 questions worth 6 marks each for a total of 30 marks.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

A

Question 1. Find and classify the critical points of the function $f(x,y) = x^3 + y^2 - 2xy$.

$f_x = 3x^2 - 2y$

$f_y = 2y - 2x = 2(y-x)$

so $f_y = 0$ if $y = x$, then $f_x = 0 \Rightarrow 3x^2 - 2x = 0$

$x(3x-2) = 0$

so $x = 0$ or $x = 2/3$

$\therefore$ there are 2 critical points $(0,0)$ and $(2/3, 2/3)$

$f_{xx} = 6x$, $f_{xy} = -2$, $f_{yy} = 2$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 12x - 4$

$D(0,0) = 12(0) - 4 = -4 < 0 \Rightarrow$ $\boxed{(0,0) \text{ is a saddle point}}$

$D(2/3, 2/3) = 12(2/3) - 4 = 4 > 0$

$f_{xx}(2/2, 2/3) = 6(2/3) = 4 > 0 \Rightarrow$ $\boxed{(2/3, 2/3) \text{ is a local min}}$

A

Question 2. Use Lagrange Multipliers to find the absolute maximum and minimum values of the function $f(x, y) = (x + y)^2$ subject to the constraint $x^2 + y^2 = 2$.

$$\nabla f = \lambda \nabla g \Rightarrow \quad 2(x+y) = \lambda(2x) \Rightarrow \quad x+y = \lambda x$$

$$2(x+y) = \lambda(2y) \Rightarrow \quad x+y = \lambda y$$

either $\lambda = 0$ or $x = y$

if $\lambda = 0$, $x+y=0 \Rightarrow x = -y$

then $x^2+y^2 = 2 \Rightarrow x = \pm 1 \Rightarrow y = \mp 1$

So 2 points $(1,-1)$ and $(-1,1)$

if $x = y$ $x^2+y^2=0 \Rightarrow x = \pm 1 \Rightarrow y = \pm 1$

2 more points $(1,1)$ and $(-1,-1)$

$f(1,1) = f(-1,-1) = 4$ max

$f(-1,1) = f(1,-1) = 0$ min

So $\boxed{\text{max value is } 4}$ and $\boxed{\text{min is } 0}$

A

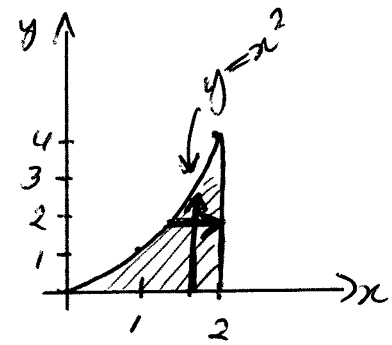
Question 3. Evaluate the following integral

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{2}{1+x^3}\, dx\, dy\,.$$

have to switch the order of integration

x goes $\sqrt{y}$ to 2

y goes 0 to 4



then we'll have y goes from 0 to x^2 and then x from 0 to 2

so $$\int_0^4 \int_{\sqrt{y}}^2 \frac{2}{1+x^3}\, dx\,dy = \int_0^2 \int_0^{x^2} \frac{2}{1+x^3}\, dy\,dx$$

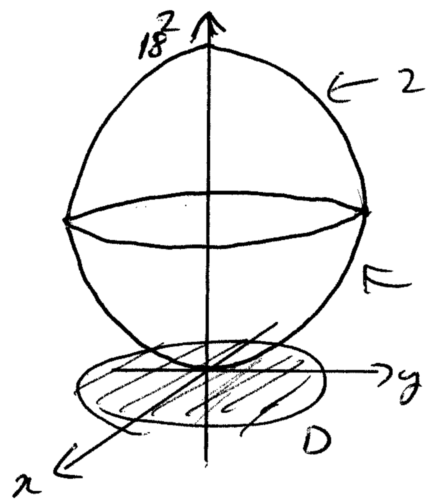
$$= \int_0^2 \frac{2x^2}{1+x^3}\, dx$$

$$= \frac{2}{3} \ln(1+x^3) \Big|_0^2$$

$$= \boxed{\frac{2}{3} \ln 9} \simeq \boxed{1.4648}$$

A

Question 4. Find the volume of the solid enclosed by the paraboloids $z = 18 - x^2 - y^2$ and $z = x^2 + y^2$.



$\leftarrow z = 18 - x^2 - y^2 = 18 - r^2$

$\leftarrow z = x^2 + y^2 = r^2$

intersection $18 - x^2 - y^2 = x^2 + y^2$

$x^2 + y^2 = 9$

So D is disk of radius 3

use cylindrical coordinates

Volume $V = \int_0^{2\pi} \int_0^3 \int_{r^2}^{18-r^2} r \, dz \, dr \, d\theta$

$= 2\pi \int_0^3 r (18 - r^2 - r^2) dr$

$= 2\pi \int_0^3 (18r - 2r^3) dr$

$= 2\pi \left(9r^2 - \frac{1}{2} r^4 \Big|_0^3\right)$

$= 2\pi \left(81 - \frac{81}{2}\right)$

$= \boxed{81\pi} \simeq \boxed{245.5}$

A

Question 5. Find the mass of the ring-like solid bounded between the spheres of radius 2 and 4 and between $\pi/4 \leq \phi \leq 3\pi/4$ if the density at any point a distance ρ from the origin is $1/\rho$.

Use spherical coordinates

Mass $$m = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_2^4 \left(\frac{1}{\rho}\right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= (2\pi)\left(\int_{\pi/4}^{3\pi/4} \sin\phi \, d\phi\right)\left(\int_2^4 \rho \, d\rho\right)$$

$$= (2\pi)\left(-\cos\phi \Big|_{\pi/4}^{3\pi/4}\right)\left(\frac{1}{2}\rho^2 \Big|_2^4\right)$$

$$= (2\pi)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)(16-4)$$

$$= (2\pi)(\sqrt{2})(6)$$

$$= \boxed{12\sqrt{2}\,\pi} \simeq \boxed{53.31}$$

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B

Question 1. Find and classify the critical points of the function $f(x, y) = 2x^3 + y^2 - 2xy$.

$f_x = 6x^2 - 2y$

$f_y = 2y - 2x = 2(y - x)$

so $f_y = 0 \Rightarrow y = x$

then $f_x = 0 \Rightarrow 6x^2 - 2x = 2x(3x - 1) = 0$

So $x = 0$ or $x = 1/3$

$\therefore$ there are 2 critical points $(0,0)$ and $(1/3, 1/3)$

$f_{xx} = 12x, \quad f_{xy} = -2, \quad f_{yy} = 2$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 24x - 4$

$D(0,0) = 24(0) - 4 = -4 < 0 \Rightarrow$ $\boxed{(0,0) \text{ is a saddle pt}}$

$D(1/3, 1/3) = 24(1/3) - 4 = 4 > 0$

$f_{xx}(1/3, 1/3) = 12(1/3) = 4 > 0 \Rightarrow$ $\boxed{(1/3, 1/3) \text{ is a local min}}$

B

Question 2. Use Lagrange Multipliers to find the absolute maximum and minimum values of the function $f(x,y)=(x+y)^2$ subject to the constraint $x^2+y^2=8$.

$\nabla f = \lambda \nabla g \Rightarrow$ $2(x+y) = 2\lambda x \Rightarrow (x+y) = \lambda x$
$2(x+y) = 2\lambda y \Rightarrow (x+y) = \lambda y$

so either $\lambda = 0$ or $x = y$

if $\lambda = 0$, $x = -y$ and so $x^2+y^2=8 \Rightarrow x = \pm 2$

thus $y = \mp 2$

giving 2 points $(2,-2)$ and $(-2,2)$

if $x = y$, $x^2+y^2=8 \Rightarrow x^2 = \pm 2 \Rightarrow y = \pm 2$

and there are 2 more points $(2,2)$ and $(-2,-2)$

$f(2,2) = f(-2,-2) = 16$ max

$f(-2,2) = f(2,-2) = 0$ min

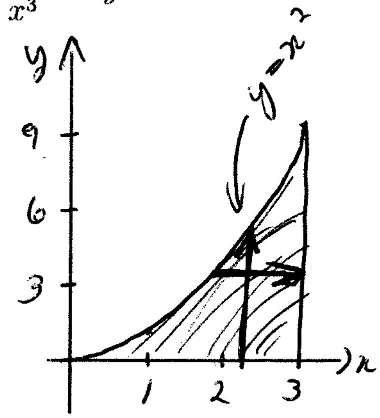
$\therefore$ $\boxed{\text{max is } 16}$ and $\boxed{\text{min is } 0}$

B

Question 3. Evaluate the following integral

$$\int_0^9 \int_{\sqrt{y}}^3 \frac{1}{1+x^3}\, dx\, dy\,.$$

have to switch the order



x goes $\sqrt{y}$ to 3

y goes 0 to 9

So then y will go from 0 to x^2

and then x goes from 0 to 3

So $\displaystyle\int_0^9 \int_{\sqrt{y}}^3 \frac{1}{1+x^3}\, dx\, dy = \int_0^3 \int_0^{x^2} \frac{1}{1+x^3}\, dy\, dx$

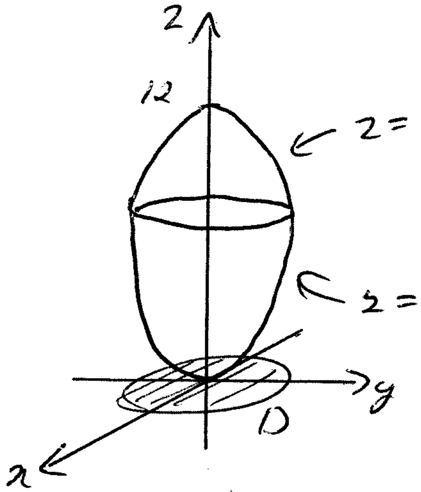
$\displaystyle = \int_0^3 \frac{x^2}{1+x^3}\, dx$

$\displaystyle = \frac{1}{3} \ln(1+x^3) \Big|_0^3$

$\displaystyle = \boxed{\frac{1}{3} \ln(28)} \simeq \boxed{1.1107}$

B

Question 4. Find the volume of the solid enclosed by the paraboloids $z = 12 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.



intersection $12 - x^2 - y^2 = 2x^2 + 2y^2$

$x^2 + y^2 = 4$

so D is disk of radius 2

use cylindrical coordinates

Volume $V = \int_0^{2\pi} \int_0^2 \int_{2r^2}^{12-r^2} r \, dz \, dr \, d\theta$

$= 2\pi \int_0^2 r(12 - r^2 - 2r^2) \, dr$

$= 2\pi \int_0^2 (12r - 3r^3) dr$

$= 2\pi \left(6r^2 - \frac{3}{4} r^4 \Big|_0^2 \right)$

$= 2\pi \left(24 - \frac{3}{4}(16) \right)$

$= \boxed{24\pi} \simeq \boxed{75.4}$

B

Question 5. Find the mass of the ring-like solid bounded between the spheres of radius 1 and 2 and between $\pi/3 \le \phi \le 2\pi/3$ if the density at any point a distance ρ from the origin is $1/\rho$.

use spherical coordinates

Mass $$M = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_1^2 \left(\frac{1}{\rho}\right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= (2\pi)\left(\int_{\pi/3}^{2\pi/3} \sin\phi \, d\phi\right)\left(\int_1^2 \rho \, d\rho\right)$$

$$= (2\pi)\left(-\cos\phi \Big|_{\pi/3}^{2\pi/3}\right)\left(\frac{1}{2}\rho^2 \Big|_1^2\right)$$

$$= (2\pi)\left(\frac{1}{2} + \frac{1}{2}\right)\left(\frac{1}{2}\right)(4-1)$$

$$= (2\pi)(1)\left(\frac{1}{2}\right)(3)$$

$$= \boxed{3\pi} \simeq \boxed{9.42}$$