① Step #1: $\frac{dy}{dx} = e^{3x} \cdot e^{2y}$ (Separable)

Step #2: $e^{-2y}\,dy = e^{3x}\,dx \Rightarrow \int e^{-2y}\,dy = \int e^{3x}\,dx$

$\boxed{-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + c} \Rightarrow \boxed{-3e^{-2y} = 2e^{3x} + \bar{c}}$ $\boxed{\text{Ans}}$

② Step #1: $\frac{dy}{dx} - \frac{1}{x}y = x\sin x$ (1st order linear with $P(x) = -\frac{1}{x}$ & $f(x) = x\sin x$)

Step #2: $IF = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x}$

Step # 3: Sol. is $\frac{1}{x} \cdot y = \int \frac{1}{x} \cdot f(x)\,dx$

$$\frac{1}{x} \cdot y = \int \frac{1}{x} \cdot x\sin x\,dx$$

$$\frac{1}{x}\,y = \int \sin x\,dx = -\cos x + c$$

$y = -x\cos x + cx$ is the gen. solution

Initial Cond.: $y\left(\frac{\pi}{2}\right) = \pi \Rightarrow \pi = -\frac{\pi}{2}\cos\frac{\pi}{2} + c \cdot \frac{\pi}{2}$

$\pi = c \cdot \frac{\pi}{2} \Rightarrow c = 2$

So the sol. of the I.V.P is $y = -x\cos x + 2x$

So $y(0) = -0 \cdot 1 + 2 \cdot 0 = \underline{\underline{0}}$ $\boxed{\text{Ans}}$

③ step #1: Verify that if it is exact

$$M = 3x^2y + e^y \quad \& \quad N = x^3 + xe^y - 2y$$

$$\frac{\partial M}{\partial y} = 3x^2 + e^y \quad \& \quad \frac{\partial N}{\partial x} = 3x^2 + e^y \quad \text{(exact)}$$

step #2: $\frac{\partial f}{\partial x} = M \Rightarrow f(x,y) = \int M(x,y)\,\partial x$

$$f(x,y) = \int \left(3x^2y + e^y\right)\partial x = x^3y + xe^y + g(y)$$

step #3: $\frac{\partial f}{\partial y} = N \Rightarrow x^3 + e^y + \frac{\partial g}{\partial y} = x^3 + xe^y - 2y$

$$\Rightarrow \frac{\partial g}{\partial y} = \frac{dg}{dy} = -2y \Rightarrow g(y) = \int -2y\,dy = -y^2.$$

$$f(x,y) = x^3y + xe^y + (-y^2)$$

step #4. Sol. is $f(x,y) = C$ or $\boxed{x^3y + xe^y - y^2 = C}$

④ step #1: $(y^2 + yx)dx = x^2dy \Rightarrow x^2\frac{dy}{dx} = y^2 + yx$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} \Rightarrow \frac{dy}{dx} \underbrace{\left(-\frac{1}{x}\right)}_{P(x)} y = \underbrace{\left(\frac{1}{x^2}\right)}_{F(x)} y^{\overset{n}{2}}$$

(Bernoulli with $n=2$, $P(x) = -\frac{1}{x}$, $F(x) = \frac{1}{x^2}$)

step #2: Let $u = y^{1-n} = y^{1-2} = y^{-1}$ and $\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$

this substitution will reduce the equation to a linear eq. in u: $\frac{du}{dx} - p(x)\,u = -F(x)$

or $\frac{du}{dx} + \frac{1}{x}u = -\frac{1}{x^2}$ (1st order linear in u)

step #3: $IF = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$

step #4: $x\,u = \int x\cdot\left(-\frac{1}{x^2}\right)dx \Rightarrow x\,u = -\int \frac{1}{x}\,dx$

Cont.d (4)

$$x \cdot u = -\ln|x| + c$$

Step #5 or $\boxed{x \cdot y^{-1} = -\ln|x| + c} \Rightarrow x = -y\ln|x| + cy$

$$\Rightarrow \boxed{x + y\ln|x| = cy}$$

(5) First you should identify what type of eq. you are dealing with.

Step #1: $\dfrac{dy}{dx} = \dfrac{y^3}{xy^2} - \dfrac{x^3}{xy^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{x^2}{y^2}$

This function can be considered to be either homogeneous or Bernoulli. [or $y' = f(y/x)$ where $f(\frac{y}{x}) = \frac{y}{x} - \left(\frac{y}{x}\right)^{-2}$
$y' - \frac{1}{x}y = -x^2 y^{-2}$

Step #2: let us the method of homogeneous.

let $y = u \cdot x$ then $\dfrac{dy}{dx} = u + x \cdot \dfrac{du}{dx}$ (This substitution will reduce the eq. to separable)

$$u + x\frac{du}{dx} = u - \frac{1}{u^2} \Rightarrow x\frac{du}{dx} = -\frac{1}{u^2} \quad \text{(separable)}$$

Step #3 $u^2\,du = -\dfrac{dx}{x} \Rightarrow \displaystyle\int u^2\,du = \int -\frac{dx}{x}$

$$\frac{1}{3}u^3 = -\ln|x| + c$$

or $\boxed{\frac{1}{3}\left(\frac{y}{x}\right)^3 = -\ln|x| + c}$ general solution

Step #4 $y(1) = 2 \Rightarrow \frac{1}{3}\left(\frac{2}{1}\right)^3 = -\ln|1| + c \Rightarrow c = \frac{8}{3}$

Sol to the I.V.P is $\frac{1}{3}\left(\frac{y}{x}\right)^3 = -\ln|x| + \frac{8}{3}$

So when $x = -1$: $\frac{1}{3}\frac{y^3}{-1} = -\ln|-1| + \frac{8}{3} \Rightarrow \frac{y^3}{-1} = 8 \Rightarrow \underline{y = -2}$

(6) $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=0}^{\infty} 2\cdot\left(\frac{2}{5}\right)^n = 2\cdot \underbrace{\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n}_{\text{geometric series with } a = 2/5 < 1}$ geometric series

$$= 2\cdot \lim_{n\to\infty} \frac{1-(2/5)^{n+1}}{1-(2/5)} =$$

$$= 2\cdot \frac{1}{1-2/5} = 2\cdot\frac{5}{5-2} = \frac{10}{3}$$

Answer

(7) $\boxed{\text{Only series (1) } \sum \frac{1}{n^5+1} \text{ converges}}$ because $\frac{1}{n^5+1} < \frac{1}{n^5}$

(2) $\sum \frac{2n-1}{3n+1}$ does not converge because $\lim_{n\to\infty} \frac{2n-1}{3n+1} = \frac{2}{3}$

(3) $\sum \frac{1}{\sqrt[3]{n}} = \sum n^{-1/3}$ does not converge because $n^{-1/3} > n^{-1}$ for $n>1$

(8) (1) $\sum \frac{(-1)^n}{5^n}$ converges ~~unconditionally~~ absolutely

(2) $\sum \frac{(-1)^n}{n+1}$ converges conditionally ← $\boxed{\text{Answer}}$

(3) $\sum \frac{(-1)^n}{n^2+1}$ converges ~~unconditionally~~ absolutely

(1) & (3) converge absolutely i.e. $\sum \left|\frac{(-1)^n}{5^n}\right| = \sum \frac{1}{5^n}$

& $\sum \left|\frac{(-1)^n}{n^2+1}\right| = \sum \frac{1}{n^2+1}$ both converge.

but $\sum \left|\frac{(-1)^n}{n+1}\right| = \sum \frac{1}{n+1}$ does not converge.

(9) Step #1 $a_n = \dfrac{n(x+2)^n}{3^{n+1}}$ & $a_{n+1} = \dfrac{(n+1)(x+2)^{n+1}}{3^{n+2}}$

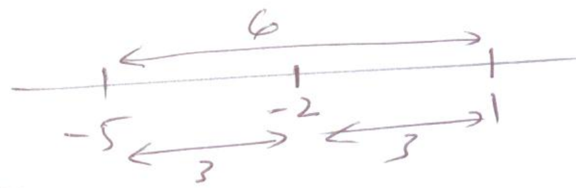
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} = \frac{(n+1)(x+2)}{3n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x+2}{3}\right| = \frac{|x+2|}{3}$$

step# 2. $\dfrac{|x+2|}{3} < 1 \Rightarrow |x+2| < 3 \Rightarrow -3 < x+2 < 3$

$\Rightarrow -3-2 < x+2-2 < 3-2 \Rightarrow -5 < x < 1$

Ans. $R = 3$



(10) step #1

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ so $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

step# 2 $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 2 = 0$

$\Rightarrow 2 - 3\lambda + \lambda^2 - 2 = 0 \Rightarrow \lambda^2 - 3\lambda = 0 \Rightarrow \begin{cases} \lambda = 0 \\ \lambda = 3 \end{cases}$

step #3 (Solve for eigenvector) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 + 2v_2 = 0$

let $v_2 = 1$ then $v_1 = -2$ so eigevector $v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is associated with $\lambda = 0$

or $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is eigenvector

$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_1 - u_2 = 0 \Rightarrow$ let $u_1 = 1$ then $u_2 = 1$

$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector associated with $\lambda = 3$

Step #4 Sol. is $\begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{0t} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

(11) $f(x) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \underbrace{\frac{f'''(1)}{3!}}_{A_3}(x-1)^3 + \cdots$

$f'(x) = \frac{1}{x} = x^{-1}$, $f''(x) = (-1)x^{-2}$, & $f'''(x) = (-1)(-2)x^{-3}$

$f'''(1) = (-1)(-2)\cdot(1)^{-3} = 2.$

Coeff. of $(x-1)^3$ is $A_3 = \frac{f'''(1)}{3!} = \frac{2}{1\cdot 2\cdot 3} = \frac{1}{3}$

(12) $f(x) = f(0) + \frac{f'(0)}{1!}x^1 + \underbrace{\frac{f''(0)}{2!}}_{A_2}x^2 + \cdots$

$f'(x) = \frac{1}{3}(1+x)^{-2/3}$ & $f''(x) = \frac{1}{3}\cdot\left(\frac{-2}{3}\right)(1+x)^{-5/3}$

$f''(0) = \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right) = \frac{-2}{9}$ so $A_2 = \frac{f''(0)}{2!} = \frac{-2/9}{2} = \frac{-1}{9}$

(13) $\int_{-1}^{1} f(x)\cdot g(x)\,dx = \int_{-1}^{1} x\cdot x^2\,dx = \int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4\Big|_{x=-1}^{1} \neq 0$

f & g are not orthogonal

$\int_{-1}^{1} f(x)\cdot h(x)\,dx = \int_{-1}^{1} x\cdot x^3\,dx = \int_{-1}^{1} x^4\,dx = \frac{1}{5}x^5\Big|_{-1}^{1} = 0$

f & h are orthogonal ← Answer

$\int_{-1}^{1} g(x)\cdot h(x)\,dx = \int_{-1}^{1} x^2\cdot x^3\,dx = \int_{-1}^{1} x^5\,dx = \frac{1}{6}x^6\Big|_{-1}^{1} \neq 0$

g & h are not orthogonal

(14) $f(x) = x$ is an odd function. Since $0 < x < L$ where $L=1$

then $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$

$$b_n = 2\int_0^1 x \sin(n\pi x)\, dx = 2\left[-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\right]_{x=0}^{1}$$

$$b_n = 2\left[-\frac{\cos(n\pi)}{n\pi} + \frac{\sin(n\pi)}{(n\pi)^2}\right] - 0 \qquad (\sin(n\pi \cdot 0) = 0)$$

$$b_n = \frac{-2\cos(n\pi)}{n\pi} \qquad (\sin(n\pi) = 0)$$

$$b_n = \frac{-2(-1)^n}{n\pi} \qquad (\cos(n\pi) = (-1)^n)$$

(15) $f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 < x \le 1 \end{cases}$ f is defined on $[-1, 1]$.

so $L = 1$.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0 \, dx + \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\cancel{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\ldots)\,dx = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} 0\,dx + \int_0^1 x}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \int_{-1}^{1} f(x).\cos(n\pi x)\, dx = \int_0^1 x\cos(n\pi x)\,dx$$

$$= \left[\frac{1}{(n\pi)^2}\cos(n\pi x) + \frac{x}{n\pi}\sin(n\pi x)\right]_{x=0}^{1} = \frac{\cos(n\pi)}{(n\pi)^2} + \frac{1}{n\pi}\sin(n\pi) - \frac{1}{(n\pi)^2}$$

$$= \frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2} \quad (\sin(n\pi) = 0 \ \& \ \cos(n\pi) = (-1)^n)$$

$$= \frac{1}{(n\pi)^2}\left[(-1)^n - 1\right]$$

Cont'd (15)

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 x\sin(n\pi x)\,dx \quad \left(\text{because } \int_{-1}^{0} f(x)\sin(n\pi x)\,dx = 0\right)$$

$$= \left[-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\right]_{x=0}^{1}$$

$$= -\frac{\cos(n\pi)}{n\pi} + \frac{\sin(n\pi)}{(n\pi)^2} - 0 = -\frac{(-1)^n}{n\pi} \quad (\text{see problem \#14})$$

(16) Step #1 aux. eqn. $m^2+5=0 \Rightarrow m = 0 \pm i\sqrt{5}$ (2 complex roots)

Step #2 Sol. is $y = c_1 e^{0x}\cos(\sqrt{5}x) + c_2 e^{0x}\sin(\sqrt{5}x)$
$= c_1\cos(\sqrt{5}x) + c_2\sin(\sqrt{5}x)$

(17) step #1: aux. eq. $m^2-6m+9=0=(m-3)^2 \Rightarrow m=3$ with multiplicity 2

homogeneous Sol. is $y_c = c_1 e^{3x} + c_2 x e^{3x}$

Step #2: $y_p(x) = Ae^x$, $y_p'(x) = Ae^x = y_p''(x)$

$y_p''(x) - 6y_p'(x) + 9y_p(x) = e^x$

$Ae^x - 6Ae^x + 9Ae^x = e^x \Rightarrow 4Ae^x = e^x \Rightarrow A = 1/4$

So $y_p(x) = \frac{1}{4}e^x$ is the particular sol.

Cont'd (17)

Gen. Sol. is $y = y_c + y_p$

$$y = \frac{1}{4}e^{x} + c_1 e^{3x} + c_2 x e^{3x}$$

(18) Step #1: aux. eq.: $m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0$

Roots are $m = -3$ & $m = 2$

$y_c = c_1 e^{-3x} + c_2 e^{2x}$ (homogeneous sol.) it is also the general sol.

$y' = -3c_1 e^{-3x} + 2c_2 e^{2x}$

Step #2: $y(0) = 1 \Rightarrow c_1 e^{0} + c_2 e^{0} = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = -1 \Rightarrow -3c_1 e^{0} + 2c_2 e^{0} = -1 \Rightarrow -3c_1 + 2c_2 = -1$

$-2c_1 - 3c_1 = -2 - 1 = -3 \Rightarrow -5c_1 = -3 \Rightarrow c_1 = 3/5$

$c_2 = 1 - c_1 = 1 - 3/5 = 2/5$

So $y = \frac{3}{5}e^{-3x} + \frac{2}{5}e^{2x}$

(19) Second order diff eqn has always 2 & linearly indep. solution Ans. (b).

(20) Step #1 It is an Euler diff. eqn. with $a = 1, b = 1,$ & $c = -9$.

aux. eq. for Euler eq is $am^2 + (b-a)m + c = 0$

or $m^2 + (1-1)m + (-9) = 0 \Rightarrow m^2 - 9 = 0 \Rightarrow$

roots are $m = 3$ & $m = -3$.

homogeneous sol. is $Y_c = c_1 x^3 + c_2 x^{-3}$.

Cont'd (20)

Step #2: Particular Sol. using the method of variation of parameters (by the way constant coeff. can't be used for this problem).

$$y'' + \frac{1}{x} y' - \frac{9}{x^2} y = \frac{1}{x^2}$$

$$y_p = u_1 \cdot y_1 + u_2 \cdot y_2$$

$$y_p = u_1 \cdot x^3 + u_2 \cdot x^{-3}$$

where
$$u_1' = \frac{\begin{vmatrix} 0 & x^{-3} \\ 1/x^2 & -3x^{-4} \end{vmatrix}}{\begin{vmatrix} x^3 & x^{-3} \\ 3x^2 & -3x^{-4} \end{vmatrix}} = \frac{-\frac{1}{x^2} \cdot x^{-3}}{-3x^{-1} - 3x^{-1}} = \frac{1}{6} x^{-4}$$

$$u_2' = \frac{\begin{vmatrix} x^3 & 0 \\ 3x^2 & 1/x^2 \end{vmatrix}}{\begin{vmatrix} x^3 & x^{-3} \\ 3x^2 & -3x^{-4} \end{vmatrix}} = \frac{x^3 \cdot 1/x^2}{-3x^{-1} - 3x^{-1}} = -\frac{1}{6} x^2$$

$$u_1 = \int \frac{1}{6} x^{-4} dx = \frac{1}{6} \frac{x^{-3}}{-3} = -\frac{1}{18} x^{-3}$$

$$u_2 = \int -\frac{1}{6} x^2 dx = -\frac{1}{6} \frac{x^3}{3} = -\frac{1}{18} x^3$$

$$y_p = -\frac{1}{18} x^{-3} \cdot x^3 + \left(-\frac{1}{18} x^3\right) \cdot x^{-3} = -\frac{1}{18} - \frac{1}{18} = -\frac{1}{9}$$

Step #3 Gen. Sol. is $y = y_p + y_c$

$$y = -\frac{1}{9} + C_1 x^3 + C_2 x^{-3}$$