# Calculus II
# MAT1322 A
# Test 2

Professor: Benoit Dionne

## True or False

**Question 1** (4 points)

For each of the following statements, determine if it is true or false (circle your answer).

| | | |
|---|---|---|
| If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. | **[TRUE]** | FALSE |
| If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges. | TRUE | **[FALSE]** |
| If the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ satisfies $a_n > 0$ and $a_{n+1} < a_n$ for all $n$, then the series converges. | TRUE | **[FALSE]** |
| If the series $\sum_{n=0}^{\infty} a_n$ converges absolutely, then the series $\sum_{n=0}^{\infty} a_n$ converges. | **[TRUE]** | FALSE |
| If the series $\sum_{n=0}^{\infty} a_n$ converges, then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely. | TRUE | **[FALSE]** |
| If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges. | TRUE | **[FALSE]** |
| Suppose that $0 \le a_n \le b_n$ and that $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges. | **[TRUE]** | FALSE |
| The geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ for all real numbers $a$ and $r$. | TRUE | **[FALSE]** |

**Solution:**

For the third statement, we also need $\lim_{n\to\infty} a_n = 0$. For the last statement, we also need $|r| < 1$.

## Multiple Choice Questions

**Question 2** (2 points)

The sum of the series $\sum_{n=0}^{\infty} \frac{30}{(n+2)(n+1)}$ is

**A**: 26 **B**: 28 **C**: 29 **D**: 30 **E**: 31 **F**: 32

**Solution:**
Using partial fractions, we have

$$\frac{30}{(n+2)(n+1)} = \frac{A}{n+2} + \frac{B}{n+1} = \frac{A(n+1) + B(n+2)}{(n+2)(n+1)}$$

Thus $30 = A(n+1) + B(n+2)$. With $n = -1$, we find $30 = B$. With $n = -2$, we find $30 = -A$. Thus

$$\frac{30}{(n+2)(n+1)} = \frac{-30}{n+2} + \frac{30}{n+1}$$

We have

$$\begin{aligned}\sum_{n=0}^{N} \frac{30}{(n+2)(n+1)} &= \sum_{n=0}^{N} \left( \frac{-30}{n+2} + \frac{30}{n+1} \right) \\ &= \left( -\frac{30}{2} + 30 \right) + \left( \frac{-30}{3} + \frac{30}{2} \right) + \left( \frac{-30}{4} + \frac{30}{3} \right) + \left( \frac{-30}{5} + \frac{30}{4} \right) \\ &+ \ldots + \left( \frac{-30}{N+2} + \frac{30}{N+1} \right) = 30 - \frac{30}{N+2}\end{aligned}$$

Since $\lim_{N\to\infty} \frac{30}{N+2} = 0$, we get

$$\sum_{n=0}^{\infty} \frac{30}{(n+2)(n+1)} = 30$$

The answer is D.

**Question 3** (**2 points**)

The sum of the series $\sum_{n=0}^{\infty} \frac{2^n - 3^{n+2}}{5^{n+1}}$ is

**A**: $-\frac{16}{3}$ **B**: $-\frac{3}{2}$ **C**: $-\frac{25}{6}$ **D**: $-\frac{9}{2}$ **E**: $-\frac{5}{2}$ **F**: $-\frac{13}{6}$

**Solution:**
We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^n - 3^{n+2}}{5^{n+1}} &= \sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}} - \sum_{n=0}^{\infty} \frac{3^{n+2}}{5^{n+1}} \\ &= \frac{1}{5} \sum_{n=0}^{\infty} \left( \frac{2}{5} \right)^n - \frac{3^2}{5} \sum_{n=0}^{\infty} \left( \frac{3}{5} \right)^n\end{aligned}$$

where the last two series are geometric series $\sum_{n=0}^{\infty} r^n$ with $r = 2/5$ and $r = 3/5$ respectively. Since $|r| < 1$, both series converge and we have

$$\sum_{n=0}^{\infty} \frac{2^n - 3^{n+2}}{5^{n+1}} = \frac{1}{5}\left(\frac{1}{1-2/5}\right) - \frac{3^2}{5}\left(\frac{1}{1-3/5}\right) = -\frac{25}{6}$$

The answer is C.

**Question 4** (**2 points**)

Find among the values below the smallest value of $N$ such that the partial sum $S_N = \sum_{n=1}^{N} \frac{1}{n^{5/2}}$ is an approximation of the series $S = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ with an error less than $10^{-3}$.

**A**: $N = 37$ **B**: $N = 47$ **C**: $N = 57$ **D**: $N = 67$ **E**: $N = 77$ **F**: $N = 87$

**Solution:**
The general term of the series $S$ is given by $a_n = f(n)$, where $f(x) = 1/x^{5/2}$. Since the function $f$ is decreasing and positive, and $\lim_{x\to\infty} f(x) = 0$, we know from the Integral Test that the series converges because $\int_1^{\infty} \frac{1}{x^{5/2}}\,\mathrm{d}x$ converges.

Moreover, we have

$$0 \leq S - S_N \leq \int_N^{\infty} \frac{1}{x^{5/2}}\,\mathrm{d}x = \left(-\frac{2}{3x^{3/2}}\right)\Bigg|_N^{\infty} = \frac{2}{3N^{3/2}}$$

We need to choose $N$ large enough such that

$$\frac{2}{3N^{3/2}} < 10^{-3} \Longrightarrow \frac{2000}{3} < N^{3/2} \Longrightarrow \left(\frac{2000}{3}\right)^{2/3} \approx 76.314 < N$$

We need to have $N \geq 77$. The answer is E.

**Question 5** (**2 points**)
You took out of your refrigerator a nice Camembert (cheese) for dinner. The initial temperature of the cheese after you took it out of the refrigerator was 4°C. The temperature of the cheese was 10°C after 15 minutes out of the refrigerator. What was the temperature of the cheese 30 minutes after you took it out of the refrigerator if the temperature of the room was 22°C?

**A**: 12.4°C **B**: 14°C **C**: 16°C **D**: 17.1°C **E**: 19.5°C **F**: 21.1°C

**Solution:**
If $y(t)$ is the temperature in Celsius of the cheese at time $t$ in minutes, we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(22 - y)$$

This is a separable equation. We have

$$\int \frac{1}{22-y}\,\mathrm{d}y = \int k\,\mathrm{d}t \Longrightarrow -\ln|22-y| = kt + C \Longrightarrow \ln|22-y| = -kt - C \qquad (1)$$
$$\Longrightarrow |22-y| = e^{-kt-C} \Longrightarrow (22-y) = Ae^{-kt} \quad , \qquad (2)$$

where $A = \pm e^{-C}$ or 0. Thus

$$y = 22 - Ae^{-kt} \quad .$$

We have

$$4 = y(0) = 22 - A \Longrightarrow A = 18$$

and

$$10 = y(15) = 22 - 18e^{-15k} \Longrightarrow \frac{2}{3} = e^{-15k} \Longrightarrow k = -\frac{1}{15}\ln\left(\frac{2}{3}\right) \approx 0.02703$$

Hence,

$$y = 22 - 18e^{-0.02703t}$$

Finally, $y(30) \approx 14$°C. The answer is B.

## Long Answer Questions

**Question 6** (**2 points**)

Use the comparison test to determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{5+3\sin(n)}{3n^3+n+4}$ converges or diverges.

**Solution:**
Since $-3 \le 3\sin(n) \le 3$ for all $n$, we have $2 \le 5+3\sin(n) \le 8$ for all $n$. Since $3n^3 < 3n^3+n+4$ for all $n > 0$, we have $\dfrac{1}{3n^3} > \dfrac{1}{3n^3+n+4}$ for all $n > 0$. Hence

$$0 < \frac{5+3\sin(n)}{3n^3+n+4} = \big(5+3\sin(n)\big)\left(\frac{1}{3n^3+n+4}\right) \le 8\left(\frac{1}{3n^3}\right) = \frac{8}{3n^3}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{8}{3n^3} = \frac{8}{3}\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges; it is a series of the form $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ with $p > 1$. Thus, from the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{5+3\sin(n)}{3n^3+n+4}$ converges.

**Question 7** (**2 points**)

Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n^2}$

**Solution:**
Let $a_n = \dfrac{3^n(x-2)^n}{n^2}$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{3^{n+1}|x-2|^{n+1}}{(n+1)^2}\right)\left(\frac{3^n|x-2|^n}{n^2}\right)^{-1} = \lim_{n\to\infty}\left(\frac{3^{n+1}|x-2|^{n+1}}{(n+1)^2}\right)\left(\frac{n^2}{3^n|x-2|^n}\right)\\ &= \lim_{n\to\infty} 3|x-2|\left(\frac{n^2}{(n+1)^2}\right) = \lim_{n\to\infty} 3|x-2|\left(\frac{n^2}{n^2+2n+1}\right)\\ &= \lim_{n\to\infty} 3|x-2|\left(\frac{1}{1+2/n+1/n^2}\right) = 3|x-2|\end{aligned}$$

Thus, the series converges if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3|x-2| < 1 \quad ;$$

namely, if $|x-2| < 1/3$. The series diverges if $|x-2| > 1/3$. So, the radius of convergence is $1/3$.

**Question 8** (**2 points**)
The power series $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{3^n\sqrt{n}}$ converges if $|x-2| < 3$ and diverges if $|x-2| > 3$. Give the interval of convergence of this power series. You must determine if the series converges at the endpoints. Justify your answer.
**Solution:**
We know that the power series will at least converge if $-3 < x-2 < 3$; namely, if $-1 < x < 5$. We only have to determine if there is convergence at $x=-1$ and $x=5$ to find the interval of convergence.

At $x=-1$, we have the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$. This is an alternating series with $\dfrac{(-1)^n}{\sqrt{n}} = (-1)^n a_n$, where $a_n = \dfrac{1}{\sqrt{n}}$. We have that

1. $a_n = \dfrac{1}{\sqrt{n}} > 0$ for all $n > 0$.
2. $a_n = \dfrac{1}{\sqrt{n}} > \dfrac{1}{\sqrt{n+1}} = a_{n+1}$ for all $n > 0$, because $\sqrt{n+1} > \sqrt{n}$ for all $n > 0$.
3. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ because $\displaystyle\lim_{n\to\infty}\sqrt{n} = \infty$

So, the alternating series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges; namely, the power series is converging at $x=-1$.

At $x=5$, we have the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. This is a series of the form $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ with $p<1$. So, the series diverges; namely, the power series diverges at $x=5$.

The interval of convergence is therefore $[-1, 5[$. The endpoint $-1$ is included but the endpoint 5 is excluded.

**Question 9** (**2 points**)

**a**) Find the MacLaurin series of the function $f(x) = \dfrac{2}{1+3x}$

**Solution:**

The geometric series is $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$. If we substitute $r = -3x$, we get $\displaystyle\sum_{n=0}^{\infty}(-3x)^n = \frac{1}{1+3x}$ for $|3x| < 1$. Thus $\displaystyle\frac{2}{1+3x} = 2\sum_{n=0}^{\infty}(-3x)^n$ for $|3x| < 1$.

**b**) Consider the series $\displaystyle g(x) = \sum_{n=0}^{\infty} \frac{nx^n}{3^n}$.

(i) Find the series for $g'(2)$.

**Solution:**

The radius of convergence of the series is 3 because

$$\lim_{n\to\infty} \left| \left( \frac{(n+1)x^{n+1}}{3^{n+1}} \right) \left( \frac{nx^n}{3^n} \right)^{-1} \right| = \frac{|x|}{3} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x|}{3} < 1$$

for $|x| < 3$. We can therefore derive the series term by term for $-3 < x < 3$. So

$$g'(x) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{nx^n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{n^2 x^{n-1}}{3^n} \Longrightarrow g'(2) = \sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{n^2}{3}\left(\frac{2}{3}\right)^{n-1}$$

(ii) Find the series for $\displaystyle\int_0^2 g(x)\,\mathrm{d}x$

**Solution:**

Since the radius of convergence is 3, we can integrate the series term by term to get

$$\int_0^2 g(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_0^2 \frac{nx^n}{3^n}\,\mathrm{d}x = \sum_{n=0}^{\infty} \left( \frac{nx^{n+1}}{(n+1)3^n} \right)\Bigg|_0^2 = \sum_{n=0}^{\infty} \frac{n2^{n+1}}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{2n}{n+1}\left(\frac{2}{3}\right)^n$$