Solution to Sample Final #1

Question 1.

The region is bounded by two paraboloids.
Their intersection is $x^2 + y^2 = 4$, which is a circle of radius 2.
So the region can be described in cylindrical coordinates:

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad r^2 \le z \le 8 - r^2.$$

So

$$\begin{aligned}
\text{volume} &= \int_0^2 \int_0^{2\pi} \int_{r^2}^{8-r^2} 1 \, dV \\
&= \int_0^2 \int_0^{2\pi} \int_{r^2}^{8-r^2} r \, dz \, d\theta \, dr \\
&= 16\pi.
\end{aligned}$$

Question 2.

Compute the curl of each vector fields. If you get a nonzero vector field, then it is not a gradient field, and vice versa.

$$\begin{aligned}
\operatorname{curl} \vec{F} &= (1-x)e^y \, \vec{k}, \\
\operatorname{curl} \vec{G} &= \vec{0}, \\
\operatorname{curl} \vec{H} &= e^y \, \vec{k}.
\end{aligned}$$

Thus, $\vec{G} = e^y \, \vec{i} + xe^y \, \vec{j}$ is the only gradient vector field.
Now find the potential:

$$\begin{aligned}
f(x,y) &= \int e^y \, dx = xe^y + C(y), \\
f_y(x,y) &= xe^y + C'(y) = xe^y, \\
C'(y) &= 0.
\end{aligned}$$

Thus, $f(x,y) = xe^y + C$ for any constant C is a potential function for $\vec{G}$.

Question 3.

S is a graph of the function $z = f(x,y) = \sqrt{x^2 + y^2}$.
The domain W, according to the question, is the ring $1 \le \sqrt{x^2 + y^2} \le 4$.
Using the formula for flux through graphs, we get:

$$\text{flux} = \int_D \vec{F} \cdot (-f_x \, \vec{i} - f_y \, \vec{j} + \vec{k}) dA,$$

where, in our case,

$$\vec{F} = 2x\sqrt{x^2+y^2}\,\vec{i} - y\sqrt{x^2+y^2}\,\vec{j} + (\sqrt{x^2+y^2} - 3y^2)\vec{k},$$

and

$$\begin{aligned} f_x &= \frac{x}{\sqrt{x^2+y^2}} \\ f_y &= \frac{y}{\sqrt{x^2+y^2}}. \end{aligned}$$

Plugging everything in, we get

$$\begin{aligned} \text{flux} &= \int_D \left(-2x^2 + y^2 + \sqrt{x^2+y^2} - 3y^2\right) dA \\ &= \int_D \left(-2(x^2+y^2) + \sqrt{x^2+y^2}\right) dA. \end{aligned}$$

In polar coordinates, this becomes

$$\begin{aligned} \text{flux} &= \int_0^{2\pi} \int_1^4 (-2r^2 + r) r \, dr \, d\theta \\ &= -213\pi. \end{aligned}$$

Question 4.

a) Use the formula for flux through spherical surfaces:

$$\text{flux} = \int_S \vec{F} \cdot \frac{\vec{r}}{\|\vec{r}\|} \, dA = \int_S \frac{\vec{r}\cdot\vec{r}}{\|\vec{r}\|} \, dA = \int_S \|\vec{r}\| \, dA,$$

where S is the sphere of radius R. Note that, in this case, $\|\vec{r}\| = R$. Thus,

$$\text{flux} = \int_S R \, dA = R \int_S dA = R \cdot (\text{area of } S) = 4\pi R^3.$$

b) Let V be the ball of radius R.
Since $\operatorname{div} \vec{F} = 1{+}1{+}1 = 3$, by the Divergence Theorem,

$$\text{flux} = \int_V 3 \, dV = 3 \cdot (\text{volume of } V) = 4\pi R^3.$$

Question 5.

a) Parametrize C: $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$, $0 \le t \le 2 * \pi$.
Then

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= -2\sin t\,\vec{i} + 2\cos t\,\vec{j}, \\ \vec{r}'(t) &= -\sin t\,\vec{i} + \cos t\,\vec{j}, \end{aligned}$$

so

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)dt\\ &= \int_0^{2\pi} (2\sin^2 t + 2\cos^2 t)dt = \int_0^{2\pi} 2\,dt\\ &= 4\pi.\end{aligned}$$

b) Upon computation, we get: $\operatorname{curl}\vec{F} = 4\vec{k}$.
By Stokes' Theorem,

$$\int_C \vec{F}\cdot\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot d\vec{A},$$

where S is the disc in xy-plane bounded by C.

Since S is inside the xy-plane, and since C is oriented as decribed in the question, the normal vector $\vec{n}$ at any point on S is $\vec{k}$.

So,

$$\begin{aligned}\int_S \operatorname{curl}\vec{F}\cdot d\vec{A} &= \int_S \vec{F}\cdot\vec{n}dA\\ &= \int_S 4\vec{k}\cdot\vec{k}dA = 4\int_S dA = 4\cdot(\text{area of } S)\\ &= 4\pi.\end{aligned}$$

Question 6.

Note that we CANNOT use the Divergence Test here.
Use the formula for flux through cylindrical surfaces.
The surface described in the question is parametrized by:

$$R = 3, \quad \pi/2 \le \theta \le 3\pi/2, \quad 0 \le z \le 1.$$

Call this surface S. Then

$$\begin{aligned}\text{flux} &= \int_{\pi/2}^{3\pi/2}\int_0^1 \vec{F}\cdot 3(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})\,dz\,d\theta\\ &= 3\int_{\pi/2}^{3\pi/2}\int_0^1 (2z\sin^2\theta\,\vec{i} - z\cos\theta\sin\theta\,\vec{j} + z^3\cos^2\theta\sin\theta\,\vec{k})\cdot(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})\,dz\,d\theta\\ &= 6\int_{\pi/2}^{3\pi/2}\int_0^1 z\sin^2\theta\cos\theta\,dz\,d\theta - 3\int_{\pi/2}^{3\pi/2}\int_0^1 z\cos\theta\sin^2\theta\,dz\,d\theta\\ &= \frac{3}{2}\int_{\pi/2}^{3\pi/2}\sin^2\theta\cos\theta\,d\theta.\end{aligned}$$

Use u substitution and get:

$$\text{flux} = -1.$$

Question 7.

a) Upon computation, we get: $\operatorname{div} \vec{F} = 0$. Therefore, by Divergence Test, $\vec{F}$ must be a curl vector field.

b) Upon computation, we DON'T get $\operatorname{curl} \vec{F} = \vec{0}$. Therefore, Curl Test will say that $\vec{F}$ is NOT a gradient vector field.

Question 8.

There is the easy way and there is the not-so-easy way.

$\boxed{\text{Not-so-easy way}}$

Note that we CANNOT use Stokes' Theorem here.

Call the straight line segment C_1 and the circular arc C_2.

C_1 is parametrized by

$$\vec{r}(t) = -3t\vec{i}, \quad 0 \le t \le 1.$$

Then

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= (1-6t)\vec{i}, \\ \vec{r}'(t) &= -3\vec{i}, \\ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= -3(1-6t), \end{aligned}$$

so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 -3(1-6t)\, dt = -3 + 9 = 6.$$

C_2 is parametrized by

$$\vec{r}(t) = 3(\cos t\, \vec{i} + \sin t\, \vec{j}), \quad \pi \le t \le 3\pi/2.$$

Then

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= (1 + 6\cos t)\vec{i} + 6\sin t\, \vec{j}, \\ \vec{r}'(t) &= 3(-\sin t\, \vec{i} + \cos t\, \vec{j}), \\ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= -3\sin t, \end{aligned}$$

so

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{\pi}^{3\pi/2} -3\sin t\, dt = 3.$$

Thus, the total line integral is

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 6 + 3 = 9.$$

$\boxed{\text{The easy way}}$

Notice that $\vec{F}$ is the gradient of $f(x, y) = x + x^2 + y^2$.

Then, by the Fundamental Theorem of Calculus for Line Integrals, we get:

$$\int_C \vec{F} \cdot d\vec{r} = f(0, -3) - f(0, 0) = (-3)^2 = 9.$$