MAT 1330, Fall 2011 Assignment 1

Due September 28 at the beginning of class.

Late assignments will **not** be accepted; **nor** will unstapled assignments.

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Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1.

A patient's metabolism will break down morphine to 60% of its original amount every day. The patient is given 8 units of morphine to raise to concentration of the medication at the end each day. Let M_t denote the concentration of morphine at time t.

(a) Write down the associated DTDS and indentify the updating function.

$$M_{t+1} = 0.6M_t + 8 \quad ; \quad f(M_t) = 0.6M_t + 8$$

(b) If the patient body contains $M_t = 5$ units of morphine at the start of the day, write down the amount of the medication at the end of the day and at the end of the next day.

End of the day: $\boxed{11}$ End of the next day: $\boxed{12.8}$

$$M_{t+1} = 0.6M_t + 8 = 0.6 \cdot 5 + 8 = 11$$

$$M_{t+2} = 0.6M_{t+1} + 8 = 0.6 \cdot 11 + 8 = 12.8$$

(c) Compose the updating function with itself and find the two step DTDS.

$$f \circ f(M_t) = f(f(M_t)) = f(0.6M_t + 8) =$$
$$= 0.6 \cdot (0.6M_t + 8) + 8 = 0.36M_t + 12.8$$

(d) Find the backwards DTDS (the inverse of the updating function) and use it to find the value at the previous time ($t = 5$), knowing that $M_6 = 11$.

Value at $t = 5$: $\boxed{5}$

$$M_{t+1} = 0.6M_t + 8$$
$$0.6M_t = M_{t+1} - 8$$
$$M_t = \frac{M_{t+1} - 8}{0.6} = f^{-1}(M_{t+1})$$
$$M_5 = \frac{M_6 - 8}{0.6} = \frac{11 - 8}{0.6} = 5$$

(e) Find the equilibrium point of the dynamical system.

Equilibrium point: $\boxed{}$

$$f(M^*) = M^*$$
$$0.6M + 8 = M$$
$$0.4M = 8$$
$$M^* = 20$$

(f) Find the general solution algebraically and use the formula to find the concentration of medication at $t = 1, 2, 3, 4$ starting at $\boxed{M_0 = 0.}$

t	1	2	3	4
M_t	8	12.8	15.68	17.408

General solution: $\boxed{\begin{aligned} M_t &= (0.6)^t M_0 + 20[1-(0.6)^t] \\ &= 20 + (0.6)^t (M_0 - 20) \end{aligned}}$

Just use the second formula, no derivation required.

$M_1 = 0.6 M_0 + 8 = 0.6 \cdot 0 + 8 = 8$

$M_2 = 0.6 M_1 + 8 = 0.6[0.6 M_0 + 8] + 8 = (0.6)^2 M_0 + 0.6 \cdot 8 + 8$

$M_3 = 0.6 M_2 + 8 = 0.6[(0.6)^2 M_0 + 0.6 \cdot 8 + 8] + 8 =$
$= (0.6)^3 M_0 + (0.6)^2 \cdot 8 + 0.6 \cdot 8 + 8$

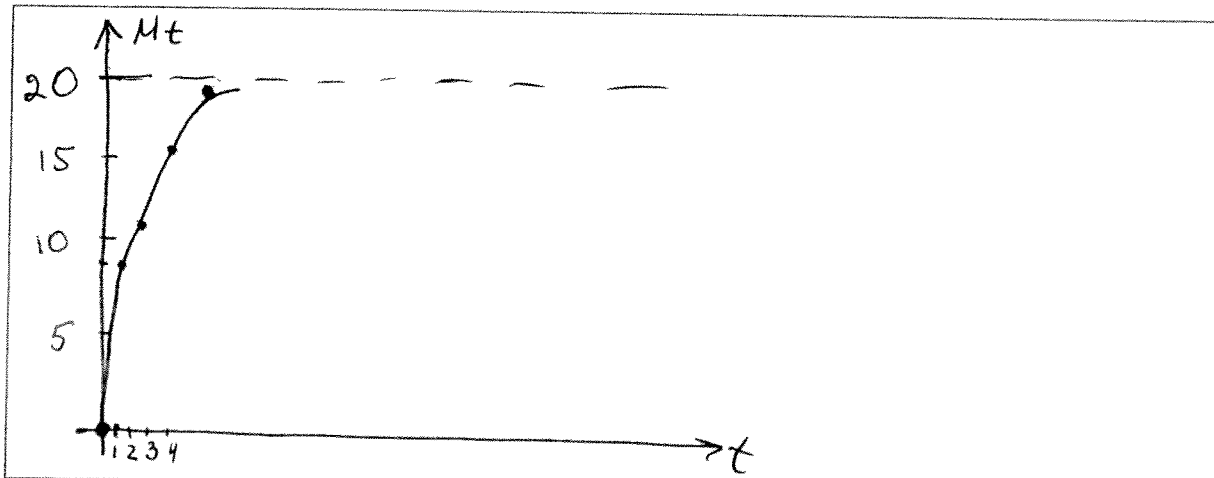
$\vdots$

$M_t = (0.6)^t M_0 + (0.6)^{t-1} \cdot 8 + (0.6)^{t-2} \cdot 8 + \dots \dots 0.6 \cdot 8 + 8 =$
$= (0.6)^t M_0 + 8[(0.6)^{t-1} + (0.6)^{t-2} + \dots + 1] =$
$= (0.6)^t M_0 + 8\left[\frac{1-(0.6)^t}{1-0.6}\right] = (0.6)^t M_0 + 20[1-(0.6)^t]$

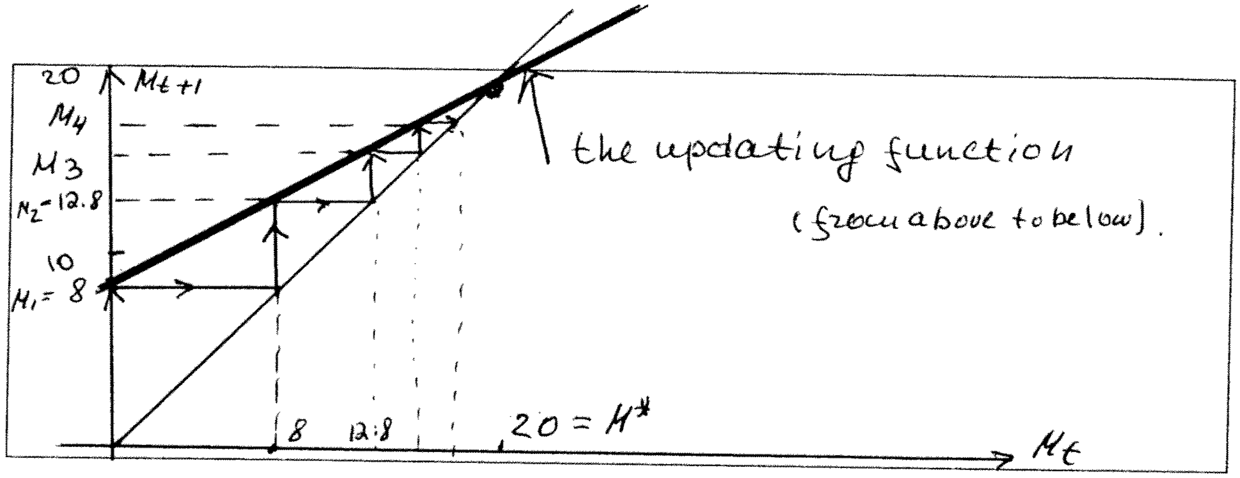
$M_1 = 8, \quad M_2 = 20[1-(0.6)^2] = 12.8$

$M_3 = 20[1-(0.6)^3] = 15.68 \quad ; \quad M_4 = 20[1-(0.6)^4] = 17.408$

(g) Draw the solution of the DTDS with the four iterations obtained in (f).



(i) Draw the cobweb diagram of the DTDS starting with $M_0 = 0$ (four iterations are enough).



(j) Determine the stability of the equilibrium point using the cobweb diagram.

Answer: Stable

QUESTION 2. For a bacteria colony that takes 3 hours to double in size, it took the colony 12 hours to reach the size of 2,000,000 bacteria. How long did it take to reach 1,000,000?

Answer: 9

$\rightarrow$ ~~bt~~ $b_{t+1} = r b_t$

$b_t = r^t b_0$

$2 b_0 = r^3 b_0$

$2 = r^3, \; r = \sqrt[3]{2} = 2^{\frac{1}{3}}$

$\rightarrow 2 \cdot 10^6 = 2^{\frac{t}{3}} \cdot b_0 \Big|_{t=12} = 2^{\frac{12}{3}} \cdot b_0 = 2^4 \cdot b_0 \Rightarrow$

$b_0 = \frac{2 \cdot 10^6}{2^4} = 125\,000$

$\rightarrow 10^6 = 2^{\frac{t}{3}} \cdot b_0 = 2^{\frac{t}{3}} \cdot 125\,000$

$8 = 2^{\frac{t}{3}} \quad \Rightarrow \quad t = 9$ hours

QUESTION 3. Consider an experiment where salt crystals are grown in a super-saturated solution. The mass of a crystal grows over 24 hours based on the updating function

$$m_{t+1} = f(m_t) = 1.5 m_t.$$

Given that the crystal originally has a mass of 10 grams, express the solution to this system, both in a table (four calculations are enough), graphically and as a formula.

$m_0 = 10$

$m_1 = 1.5 \cdot m_0 = 15$

$m_2 = 1.5 \cdot m_1 = 1.5 \cdot 15 = 22.5$

$m_3 = 1.5 m_2 = 1.5 \cdot 22.5 = 33.75$

$\vdots$

$m_t = (1.5)^t m_0.$

t	0	1	2	3
m_t	10	15	22.5	33.75

QUESTION 4. A population grows according to the formula $b_{n+1} = \dfrac{3b_n}{1 + 0.2b_n}$.

(a) Find all the equilibrium points of the DTDS.

Equilibrium points: $\boxed{b_1^* = 0,\ b_2^* = 10}$

$b_{n+1} = f(b_n) = \dfrac{3b_n}{1+0.2b_n}$

$b^* = f(b^*)$

$\dfrac{3b}{1+0.2b} = b$

$3b = b + 0.2b^2$

$2b - 0.2b^2 = 0$

$b_1^* = 0$

$b_2^* = 10$

(b) Assume that we start with $b_0 = 100$. What will be the size of the population after the 1st, 2nd, and 3rd year.

$$b_1 = \frac{3b_0}{1+0.2\,b_0} = \frac{3\cdot 100}{1+0.2\cdot 100} \approx 14 \quad ; \quad b_3 = \frac{3\,b_2}{1+0.2\,b_2} \approx 10.4$$

$$b_2 = \frac{3\,b_1}{1+0.2\,b_1} = \frac{3\cdot 14}{1+0.2\cdot 14} \approx 11.11$$

(c) Assume that we start with $b_0 = 10$. What will be the size of the population b_1, b_2, b_3?

$b_0 = b_1 = b_2 = \cdots = 10$ ⇐ the carrying capacity

(d) Assume that we start with $b_0 = 3$. What will be the size of the population b_1, b_2, b_3?

$$b_1 = \frac{3b_0}{1+0.2\,b_0} = \frac{9}{1.6} \approx 5.6 \qquad b_3 = \frac{3\,b_2}{1+0.2\,b_2} \approx 9.$$

$$b_2 = \frac{3\,b_1}{1+0.2\,b_1} \approx 8$$

(e) Sketch the solution curves obtained in $(b), (c), (d)$ in the same tb_t-plane. Give short comments about the above results (three sentences are enough).