INNER PRODUCT, LENGTH, and ORTHOGONALITY

Definition: Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

be two vectors in R^n. Then the inner product of u and v is

$$\begin{aligned} u \cdot v &= u^T v = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\ &= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n. \end{aligned}$$

Example: Let $u = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $v = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Then

$$\begin{aligned} u \cdot v &= u^T v = [3 \;\; -1 \;\; -5] \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \\ &= 3 \cdot 6 + (-1) \cdot (-2) + (-5) \cdot 3 \\ &= 18 + 2 - 15 = 5, \text{ and} \end{aligned}$$

$$
\begin{aligned}
v \cdot u &= v^T u = [6 \ \ -2 \ \ 3] \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \\
&= 6 \cdot 3 + (-2)(-1) + 3 \cdot (-5) \\
&= 18 + 2 - 15 = 5.
\end{aligned}
$$

So, $u \cdot v = v \cdot u$

Properties of the inner (dot) product

Let u, v, and w be vectors in R^n and c be a scalar. Then,

1) $u \cdot v = v \cdot u$

2) $(u + v) \cdot w = u \cdot w + v \cdot w$

3) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

4) $u \cdot u \geq 0$ and $u \cdot u = 0 \Longleftrightarrow u = 0$.

Definition (the length of a vector):
If $v = (v_1, v_2, ..., v_n)$ in R^n, then the length (or norm) of v is defined by

$$||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Then, $||v||^2 = v \cdot v$.

A vector whose length is one unit is called a **unit vector**.

Example: Let $v = (2, -3, 1)$. Find the length of v and a unit vector in the direction of v.

Solution:

$$\begin{aligned} ||v||^2 &= v \cdot v = 2^2 + (-3)^2 + 1^2 = 14, \\ ||v|| &= \sqrt{14} \end{aligned}$$

A unit vector in the direction of v is

$$\frac{1}{\sqrt{14}}(2, -3, 1) = \left(\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) = u.$$

The process of creating u from v is called **normalizing** v.

Example: Let $W = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}$.

Find a unit vector v which is a basis for W.

Solution: let $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Then,

$||u|| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$.

$$v = \frac{1}{\sqrt{6}} u = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

Another unit vector is

$$-v = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}.$$

Definition (Distance between u and v): Let u, v in R^n. Then the distance between u and v is the length of the vector $u - v$. That is

$$dist(u, v) = ||u - v||.$$

Example: Let $u = (1, 2, -1)$ and $v = (2, 1, 1)$ in R^3. Then

$$\begin{aligned} dist(u, v) &= ||u - v|| = ||(-1, 1, -2)|| \\ &= \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}. \end{aligned}$$

Definition (Orthogonal vectors): Let u and v be two vectors in R^n. Then u and v are orthogonal to each other ($u \perp v$) if $u \cdot v = 0$. Note that zero vector is orthogonal to every vector.

Example (The Phythagorean Theorem):
Let u and v in R^n. Show that

$$u \perp v \Longleftrightarrow ||u+v||^2 = ||u||^2 + ||v||^2.$$

Solution:

$$\begin{aligned} ||u+v||^2 &= (u+v)\cdot(u+v) \\ ||u+v||^2 &= ||u||^2 + ||v||^2 + 2u\cdot v \quad (*) \end{aligned}$$

If $u \perp v$, then $u \cdot v = 0$, and (*) gives

$$||u+v||^2 = ||u||^2 + ||v||^2.$$

If $||u+v||^2 = ||u||^2 + ||v||^2$, then (*) gives that

$$u \cdot v = 0.$$

Exercise: Let u and v in R^n. Show that

$$u \perp v \Longleftrightarrow ||u+v||^2 = ||u-v||^2.$$

Hint: Consider the equations
$||u+v||^2 = ||u||^2 + ||v||^2 + 2u\cdot v$ and
$||u-v||^2 = ||u||^2 + ||v||^2 - 2u\cdot v$

Question: Does

$$||u + v + z||^2 = ||u||^2 + ||v||^2 + ||z||^2$$

imply that $\{u, v, z\}$ is an orthogonal set?

Answer: No. Consider the vectors

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; v = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ and } z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then,

$$||u + v + z||^2 = ||[2, 1, 1]^T||^2 = 6,$$

$$||u||^2 = ||v||^2 = ||z||^2 = 2, \text{ and thus}$$

$$||u + v + z||^2 = ||u||^2 + ||v||^2 + ||z||^2.$$

But $\{u, v, z\}$ is not an orthogonal set.

Definition: A set of non-zero vectors S is called an orthogonal set if each vector in S is orthogonal to other vectors in S, i.e,

$$S = \{v_1, v_2, ..., v_p\} \text{ is an orthogonal set}$$

$$\iff v_i \cdot v_j = 0 \text{ if } i \neq j.$$

- An orthogonal set in which each vector has length 1 is called an orthonormal set.

- A basis consisting of orthogonal vectors is called an orthogonal basis.

- A basis consisting of orthonormal vectors is called an orthonormal basis.

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

is an orthonormal basis for R^3.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for R^3.

$$\left\{ \begin{bmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{12} \\ 2/\sqrt{12} \\ 2/\sqrt{12} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} \right\}$$

is an orthonormal basis for R^3.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}$$

is an orthogonal set in R^3.

$$\left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}$$

is an orthonormal set in R^3.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal basis for R^3.

$$\left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \right\}$$

is an orthonormal basis for R^3.

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal set in R^4.

$$\left\{ \begin{bmatrix} 2/\sqrt{24} \\ 2/\sqrt{24} \\ 4/\sqrt{24} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

is an orthonormal set in R^4.

Theorem: If $S = \{u_1, u_2, ..., u_p\}$ is an orthogonal set of <u>nonzero</u> vectors in R^n, then S is linearly independent.

Proof: Let $0 = c_1u_1 + c_2u_2 + \cdots + c_pu_p$ for some scalars $c_1, c_2, ..., c_p$. Then,

$$0 \cdot u_1 = (c_1u_1 + c_2u_2 + \cdots + c_pu_p) \cdot u_1$$

$$0 = c_1u_1 \cdot u_1 + c_2u_2 \cdot u_1 + \cdots + c_pu_p \cdot u_1,$$

$$0 = c_1u_1 \cdot u_1 + 0 + \cdots + 0,$$

since $u_i \cdot u_j = 0$ if $i \neq j$.

$0 = c_1||u_1||^2$which gives $c_1 = 0$

since $u_1 \neq 0$.

Similarly, we can show that $c_2 = 0$,...,$c_p = 0$.

Thus, S is linearly independent.

Theorem: Let $\{u_1, u_2, ..., u_n\}$ be an orthogonal basis for R^n. Then for each $x \in R^n$,

$$x = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n \text{ where}$$

$$c_i = \frac{x \cdot u_i}{u_i \cdot u_i}, \quad i = 1, 2, ..., n.$$

Proof: For a fixed i, $1 \leq i \leq n$,

$$x \cdot u_i = (c_1 u_1 + c_2 u_2 + \cdots + c_n u_n) \cdot u_i = c_i(u_i \cdot u_i),$$

which gives

$$c_i = \frac{x \cdot u_i}{u_i \cdot u_i}$$

(since $u_i \neq 0$, $u_i \cdot u_i \neq 0$)

Example: Let

$$u_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

and $S = \{u_1, \ u_2, \ u_3\}$. Express $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in S.

Solution: Clearly $u_i \cdot u_j = 0$, when $i \neq j$. So, S is an orthogonal basis for R^3.

$$\begin{aligned} c_1 &= \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{-1+4+6}{1+4+4} = \frac{9}{9} = 1, \\ c_2 &= \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{2-2+6}{4+1+4} = \frac{6}{9} = \frac{2}{3}, \\ c_3 &= \frac{x \cdot u_3}{u_3 \cdot u_3} = \frac{2+4-3}{4+4+1} = \frac{3}{9} = \frac{1}{3}. \end{aligned}$$

$$\begin{aligned} x &= c_1u_1 + c_2u_2 + c_3u_3, \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} &= 1 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}. \end{aligned}$$