Assignment 1: Solutions

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

   (a) $\underbrace{(p \wedge \underbrace{(\neg(\neg p \vee q))}_{\mathbf{a}})}_{\mathbf{b}} \vee \underbrace{(p \wedge q)}_{\mathbf{c}}$

   Solution: Contingency.

p	q	$\neg p$	$\neg p \vee q$	$\overbrace{\neg(\neg p \vee q)}^{\mathbf{a}}$	$\overbrace{p \wedge \mathbf{a}}^{\mathbf{b}}$	$\overbrace{p \wedge q}^{\mathbf{c}}$	$\mathbf{b} \vee \mathbf{c}$
T	T	F	T	F	F	T	T
T	F	F	F	T	T	F	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F

   (b) $(\underbrace{(p \to r)}_{\mathbf{a}} \vee \underbrace{(q \to r)}_{\mathbf{b}}) \to \underbrace{((p \vee q) \to r)}_{\mathbf{c}}$

   Solution: Contingency.

p	q	r	$\overbrace{p \to r}^{\mathbf{a}}$	$\overbrace{q \to r}^{\mathbf{b}}$	$\mathbf{a} \vee \mathbf{b}$	$p \vee q$	$\overbrace{(p \vee q) \to r}^{\mathbf{c}}$	$(\mathbf{a} \vee \mathbf{b}) \to \mathbf{c}$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	T	T	F	F
F	T	F	T	F	T	T	F	F
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

   (a) $p \to (q \to r) \ \equiv\ q \to (\neg p \lor r)$

   Solution: Valid.

$$
\begin{array}{lll}
 & p \to (q \to r) & \\
\equiv & \neg p \lor (q \to r) & \text{law for conditional} \\
\equiv & \neg p \lor (\neg q \lor r) & \text{law for conditional} \\
\equiv & (\neg p \lor \neg q) \lor r & \text{associativity} \\
\equiv & (\neg q \lor \neg p) \lor r & \text{commutativity} \\
\equiv & \neg q \lor (\neg p \lor r) & \text{associativity} \\
\equiv & q \to (\neg p \lor r) & \text{law for conditional}
\end{array}
$$

   (b) $(p \to q) \land (p \to r) \ \equiv\ (p \to (q \land r))$

   Solution: Valid.

$$
\begin{array}{lll}
 & (p \to q) \land (p \to r) & \\
\equiv & (\neg p \lor q) \land (\neg p \lor r) & \text{law for conditional} \\
\equiv & \neg p \lor (q \land r) & \text{distributivity} \\
\equiv & (p \to (q \land r)) & \text{law for conditional}
\end{array}
$$

3. Write the following statements in predicate form, using logical operators $\wedge$, $\vee$, $\neg$, and quantifiers $\forall$, $\exists$. Below $\mathbb{Z}^+$ denotes all positive integers $\{1, 2, 3, \ldots\}$.

   (a) The equation $x^3 + y^3 = z^3$ has no solutions $x, y, z \in \mathbb{Z}^+$.

   Solution:

   $$\neg\Big(\exists\, x, y, z \in \mathbb{Z}^+ \big(x^3 + y^3 = z^3\big)\Big)$$

   or equivalently

   $$\forall\, x, y, z \in \mathbb{Z}^+ \big(x^3 + y^3 \neq z^3\big)$$

   (b) There is no greatest positive integer.
   Here the Universe of Discourses for x and y is $\mathbb{Z}^+$.

   Solution:

   $$\forall x \exists y \ \big(x < y\big)$$

4. Let the Universe of Discourse for x be the set of all students in this class and for y be the set of all countries in the world. Let $P(x, y)$ denote student x has visited country y and $Q(x, y)$ denote student x has a friend in country y. Express each of the following using logical operations and quantifiers, and the propositional functions $P(x, y)$ and $Q(x, y)$.

   (a) Nobody in this class has visited a country in which they did not have a friend.

   Solution:

   $$\neg\Big(\exists x \exists y \big(P(x, y) \wedge \neg Q(x, y)\big)\Big)$$

   or equivalently

   $$\forall x \forall y \big(P(x, y) \rightarrow Q(x, y)\big)$$

   (b) There is a country in which nobody in this class has a friend.

   Solution:

   $$\exists y \forall x \big(\neg Q(x, y)\big)$$

   (c) Everyone in this class has visited every country in which they have a friend.

   Solution:

   $$\forall x \forall y \big(Q(x, y) \rightarrow P(x, y)\big)$$

5. Negate the following statements and transform the negation so that negation symbols immediately precede predicates.

   (a) $\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)$

   Solution:

$$\begin{aligned} & \neg\Big(\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)\Big) \\ \equiv\ & \neg\Big(\exists x \exists y P(x,y)\Big) \vee \neg\Big(\forall x \forall y Q(x,y)\Big) \\ \equiv\ & \Big(\forall x \forall y \neg P(x,y)\Big) \vee \Big(\exists x \exists y \neg Q(x,y)\Big) \end{aligned}$$

   (b) $\exists x \exists y (Q(x,y) \leftrightarrow Q(y,x))$

   Solution:

$$\begin{aligned} & \neg\Big(\exists x \exists y \big(Q(x,y) \leftrightarrow Q(y,x)\big)\Big) \\ \equiv\ & \forall x \forall y \Big(\neg\big(Q(x,y) \leftrightarrow Q(y,x)\big)\Big) \\ \equiv\ & \forall x \forall y \neg\Big(\big(Q(x,y) \rightarrow Q(y,x)\big) \wedge \big(Q(y,x) \rightarrow Q(x,y)\big)\Big) \\ \equiv\ & \forall x \forall y \Big(\neg\big(Q(x,y) \rightarrow Q(y,x)\big) \vee \neg\big(Q(y,x) \rightarrow Q(x,y)\big)\Big) \\ \equiv\ & \forall x \forall y \Big(\neg\big(\neg Q(x,y) \vee Q(y,x)\big) \vee \neg\big(\neg Q(y,x) \vee Q(x,y)\big)\Big) \\ \equiv\ & \forall x \forall y \Big(\big(Q(x,y) \wedge \neg Q(y,x)\big) \vee \big(Q(y,x) \wedge \neg Q(x,y)\big)\Big) \end{aligned}$$

   (c) $\forall y \exists x \exists z (T(x,y,z) \vee Q(x,y))$

   Solution:

$$\begin{aligned} & \neg\Big(\forall y \exists x \exists z \big(T(x,y,z) \vee Q(x,y)\big)\Big) \\ \equiv\ & \exists y \forall x \forall z \neg\big(T(x,y,z) \vee Q(x,y)\big) \\ \equiv\ & \exists y \forall x \forall z \big(\neg T(x,y,z) \wedge \neg Q(x,y)\big) \end{aligned}$$

6. Let P and Q be predicates on the set S, where S has two elements, say, $S = \{a, b\}$. Then the statement $\forall x P(x)$ can also be written in full detail as $P(a) \wedge P(b)$. Rewrite each of the statements below in a similar fashion, using P, Q, and logical operators, but without using quantifiers.

   (a) $\forall x \forall y (P(x) \vee Q(y))$

   Solution:

$$\begin{aligned} & \forall x \Big(P(x) \vee \big(Q(a) \wedge Q(b) \big) \Big) \\ \equiv \quad & \big(P(a) \wedge P(b) \big) \vee \big(Q(a) \wedge Q(b) \big) \end{aligned}$$

   (b) $\exists x P(x) \wedge \exists x Q(x)$

   Solution:

$$\big(P(a) \vee P(b) \big) \wedge \big(Q(a) \vee Q(b) \big)$$

   (c) $\exists x \exists y (P(x) \wedge Q(y))$

   Solution:

$$\begin{aligned} & \exists y \Big(\big(P(a) \vee P(b) \big) \wedge Q(y) \Big) \\ \equiv \quad & \big(P(a) \vee P(b) \big) \wedge \big(Q(a) \vee Q(b) \big) \end{aligned}$$

   (d) $\forall x \exists y (P(x) \wedge Q(y))$

   Solution:

$$\begin{aligned} & \forall x \Big(P(x) \wedge \big(Q(a) \vee Q(b) \big) \Big) \\ \equiv \quad & \Big(P(a) \wedge \big(Q(a) \vee Q(b) \big) \Big) \wedge \Big(P(b) \wedge \big(Q(a) \vee Q(b) \big) \Big) \\ \equiv \quad & \Big(\big(P(a) \wedge Q(a) \big) \vee \big(P(a) \wedge Q(b) \big) \Big) \wedge \Big(\big(P(b) \wedge Q(a) \big) \vee \big(P(b) \wedge Q(b) \big) \Big) \end{aligned}$$

7. For each of the following equivalences, determine if it is valid for all predicates P and Q. If yes then give a full explanation. If not then provide a counterexample.

(a) $\exists x(P(x) \vee Q(x)) \;\equiv\; \exists x P(x) \vee \exists x Q(x)$

Solution: *True.*

If the LHS is *True* then there is a constant, say, a such that (i) $P(a) = T$, or (ii) $Q(a) = T$. In case (i) $\exists x P(x)$ is *True*, while in case (ii) $\exists x Q(x)$ is *True*, so that in both cases the RHS is *True*,

If the RHS is *True*, then (i) there is a constant, say, a such that $P(a) = T$, so that the LHS is also True with $x = a$, or (ii) there is a constant, say, b such that $Q(b) = T$, so that the LHS is also True with $x = b$.

(b) $\exists x(P(x) \wedge Q(x)) \;\equiv\; \exists x P(x) \wedge \exists x Q(x)$

Solution: *False.*

Suppose $P(a)$ is *True* for some constant a, and *False* for all constants $b \neq a$. Suppose also $Q(b)$ is *True* for some constant b, and *False* for all constants $a \neq b$. Then RHS is *True* and LHS is *False*.

(c) $\forall x \forall y(P(x) \wedge Q(y)) \;\equiv\; \forall x P(x) \wedge \forall x Q(x)$

Solution: *True.*

If the RHS is *True* then P is always *True* and Q is always *True*. Then the LHS is also always *True*.

If the RHS is *False* then there is a constant, say a, such that $P(a) = F$ and there is a constant, say b, such that $Q(b) = F$. Then $P(a) \wedge P(b)$ is *false*, so the LHS is also *False*.

(d) $\forall x \forall y(P(x) \vee Q(y)) \;\equiv\; \forall x P(x) \vee \forall y Q(y)$

Solution: *True.*

If the RHS is *True* then we must consider two cases: (i): P is always True or (ii) Q is always True. In either case the LHS is True also.

If on the other hand the RHS is *False* then there is a constant, say a, such that $P(a) = F$ and there is a constant, say b, such that $Q(b) = F$. Then the LHS is seen to be False also, namely by picking $x = a$ and $y = b$.