# MATRIX OPERATIONS

An $m \times n$ matrix $A$ is a rectangular array of numbers with $m$ rows and $n$ columns.

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3}, B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 8 & 6 \end{bmatrix}_{3\times 2}, \text{ and}$$

$$C = \begin{bmatrix} 3 & 5 & 0 \\ -1 & 2 & 1 \end{bmatrix}_{2\times 3}.$$

Then

$$A + C = \begin{bmatrix} 1+3 & 2+5 & 3+0 \\ 4-1 & 5+2 & 7+1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 3 \\ 3 & 7 & 8 \end{bmatrix}_{2\times 3},$$

$A + B =$ can not be done.

If $A$ and $B$ are matrices of the same size, their sum $A+B$ is a matrix of the same size formed by adding corresponding entries in $A$ and $B$.

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3} \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 8 & 6 \end{bmatrix}_{3\times 2} \\
&= \begin{bmatrix} 1\cdot 4 + 2\cdot 1 + 3\cdot 8 & 1\cdot 0 + 2\cdot 3 + 3\cdot 6 \\ 4\cdot 4 + 5\cdot 1 + 7\cdot 8 & 4\cdot 0 + 5\cdot 3 + 7\cdot 6 \end{bmatrix} \\
&= \begin{bmatrix} 30 & 22 \\ 77 & 57 \end{bmatrix}_{2\times 2}.
\end{aligned}
$$

$$A_{2\times 3}C_{2\times 3} = \text{ can not be done.}$$

$$C_{2\times 3}A_{2\times 3} = \text{ can not be done.}$$

In general, $\boxed{A_{m\times n}B_{n\times k} = C_{m\times k}}$.

To multiply two matrices $A$ and $B$, the number of columns of $A$ must be the same as the number of rows of $B$.

**Example:** Let

$$A = \begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 2 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

$$CA = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 2 & 4 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2},$$

$$\begin{aligned} AC &= \begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 2 & 4 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}_{2\times 3} \\ &= \begin{bmatrix} 7 & 6 & -3 \\ -4 & -3 & 2 \\ 6 & 6 & -2 \end{bmatrix}_{3\times 3}. \end{aligned}$$

$AC$ and $CA$ not only different, but also they have different sizes.

$I_{3\times 3} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix of size $3 \times 3$.

For any $m \geq 1$, $I_3 A_{3\times m} = A_{3\times m}$, and

for any $n \geq 1$, $B_{n\times 3} I_3 = B_{n\times 3}$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not an identity matrix.

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is also a zero matrix.

An $m \times n$ matrix whose entries are all zero is a <u>zero matrix</u>.

We denote an $m \times n$ matrix $A$ as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

The entry in the $i$th row and $j$th column is called the $(i, j)$-entry of $A$. If $m = n$, then $A$ is called a **square matrix of order n**, and the entries $a_{11}$, $a_{22}, \ldots,$ $a_{nn}$ form the **main diagonal** of $A$.

The identity matrix of size $n \times n$ is a matrix such that all the main diagonal entries are 1, and all other entries are 0.

If $A$ is a matrix and $c$ is a scalar, the product $cA$ is a matrix of the same size as $A$ in which every entry is multiplied by $c$.

**Example:**

$$5 \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 \cdot (-1) & 5 \cdot 0 \\ 5 \cdot 2 & 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 10 & 15 \end{bmatrix}.$$

**Example:** Let

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

$$A + B = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix} = B + A.$$

$$AB = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

We see that $A + B = B + A$ but $AB \neq BA$.

**Example:** Let

$$A = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$AB = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, $AB = 0$ but $A \neq 0$, $B \neq 0$.

**Example:** Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

$$\left.\begin{aligned} AB &= \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \\ AC &= \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \end{aligned}\right\} \quad \begin{array}{c} AB = AC \\ B \neq C \end{array}$$

**Homework:** Let

$$A = \begin{bmatrix} 7 & 0 & -1 \\ -1 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 4 & 1 \\ 5 & -3 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 4 \\ -4 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 7 \\ -3 \end{bmatrix}.$$

Determine if each of the following matrices is defined:

$-2A$, $B + 4A$, $AC$, $CD$, $A + B$,
$3C - E$, $CB$, $EB$.

**Example:** Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Find $A^2$, $A^3$ and $A^4$.

**Solution:**

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A^3 = AA^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly,

$$A^4 = AA^3 = \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Exercise:** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $A^4$.

Make a guess for $A^n$.

**Example:** Let $A = \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix}$ and

$B = \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix}$.

For what value(s) of $k$, $AB = BA$?

**Solution:**

$$\begin{aligned} AB &= \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix} \\ &= \begin{bmatrix} 21-20 & 12-4k \\ -35+5 & -20+k \end{bmatrix} \\ &= \begin{bmatrix} 1 & 12-4k \\ -30 & -20+k \end{bmatrix} \\ BA &= \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -24 \\ 15-5k & -20+k \end{bmatrix} \end{aligned}$$

Then $AB = BA \Leftrightarrow \left\{ \begin{array}{l} 15-5k = -30 \text{ and} \\ 12-4k = -24. \end{array} \right\}$

Thus, $AB = BA \Leftrightarrow k = 9$.

**Example:** Let

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \text{ and } AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}.$$

Determine the first and second columns of $B$.

**Solution:** Let $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}.$$

This gives that

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} a - 2d \\ -2a + 5d \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right].$$

So, $a = 7$ and $d = 4$, i.e, $\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

To determine the second column of $B$ is left as an exercise.

The **transpose** of an $m \times n$ matrix $A$ is the $n \times m$ matrix whose columns are the corresponding rows of $A$, and denoted by $A^T$.

**Example:**

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & 7 & 8 \\ 0 & 1 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 3 & 6 & 0 \\ -1 & 7 & 1 \\ 2 & 8 & 5 \end{bmatrix}.$$

$$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

**Properties of Transpose:**

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(rA)^T = rA^T$$

$$(AB)^T = B^T A^T.$$

**Example:** Let

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Then,

$$B^T A^T = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 10 & -1 \end{bmatrix},$$

$$AB = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 4 & -1 \end{bmatrix}.$$

Thus, $(AB)^T = \begin{bmatrix} 2 & 4 \\ 10 & -1 \end{bmatrix}$.

So, we have $(AB)^T = B^T A^T = \begin{bmatrix} 2 & 4 \\ 10 & -1 \end{bmatrix}$.

**Example:** Let

$A = \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$. Then,

$$(AB)^T = B^T A^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 14 & 28 \\ 15 & 35 \end{bmatrix}.$$

# Inverse of a Matrix

**Definition:** Let $A$ be an $\underline{n \times n \text{ square}}$ matrix. If there is a matrix $B$ such that $AB = BA = I_n$ then $A$ is said to be **invertible** and $B$ is called the inverse of $A$.

**Theorem:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$A$ is invertible $\Leftrightarrow ad - bc \neq 0$.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $A$ is not invertible.
$\det A = ad - bc$ is called the determinant of $A$.

**Example:** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$ad - bc = 1 \cdot 4 - 3 \cdot 2 = -2 \neq 0,$$

so $A$ is invertible.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}, \quad \text{and}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

**Theorem:** If $A$ is an invertible $n \times n$ matrix, then for each $b \in R^n$, the equation $AX = b$ has the unique solution

$$X = A^{-1}b.$$

**Example:** Solve the equation

$$AX = \begin{bmatrix} -5 & -11 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

**Solution:**

$$\begin{aligned} \det A &= ad - bc \\ &= -5 \cdot 7 - 3 \cdot (-11) \\ &= -35 + 33 \\ &= -2 \neq 0, \end{aligned}$$

so $A$ is invertible, and

$$\begin{aligned} X &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & -11 \\ 3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (-1/2) \begin{bmatrix} 7 & 11 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}. \end{aligned}$$

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$$\left[\begin{array}{cc|c} -5 & -11 & 1 \\ 3 & 7 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} -15 & -33 & 3 \\ 15 & 35 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} -15 & -33 & 3 \\ 0 & 2 & 8 \end{array}\right]$$

## Elementary Matrices

An elementary matrix $E$ is a square matrix obtained by performing a single row operation on an identity matrix.

**Example:** For each of the following row operation, find the corresponding elementary matrix.

**I)** $R_2 \longleftrightarrow R_3$ **II)** $R_1' = 4R_1$ **III)** $R_2' = R_2 - 7R_1$

**Solution: I)**

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \longleftrightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1$$

**II)**

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1' = 4R_1 \quad \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

**III)**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2' = \underbrace{R_2 - 7R_1}_{a_{21}=-7} \quad \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \\ c & d \end{bmatrix},$$

$$E_2 A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 4a & 4b \\ c & d \\ e & f \end{bmatrix},$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ c - 7a & d - 7b \\ e & f \end{bmatrix}.$$

Note that the matrices $E_1A$, $E_2A$ and $E_3A$ are the matrices obtained from $A$ by performing the elementary row operations I, II and III, respectively.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

**Example:**

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longrightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that $E_1 = E_1^{-1}$.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$E_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow E_3^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Example:** Perform the following row operations on $I_2$, and write corresponding elementary matrices and their inverses.

- $R_1' = \underbrace{R_1 - 3R_2}_{a_{12}=-3 \text{ in } E_1}$ :

$$E_1 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \Longrightarrow E_1^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

- $R_2' = 4R_2$ :

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \Longrightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

- $R_1 \leftrightarrow R_2$ :

$$E_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Longrightarrow E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In general we have the followig:

| Elementary Row Operation | Corresponding Inverse Operation |
|---|---|
| $R_i \leftrightarrow R_j$ | $R_j \leftrightarrow R_i$ |
| $R_i' = cR_i$ | $R_i' = (1/c)R_i$ |
| $R_i' = R_i + cR_j$ | $R_i' = R_i - cR_j$ |

An $n \times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to $I_n$, ~~and in this case,~~

Any sequence of elementary row operations that reduces $A$ to $I_n$ also transforms $I_n$ to $A^{-1}$.

**Example:** Write $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^{-1}$ as products of elementary matrices.

**Solution:** First we compute $A^{-1}$.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \quad R_2{}' = R_2 - R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \quad R_1{}' = R_1 - R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

$$R_2{}' = R_2 - R_1 : \quad \Longrightarrow E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

$$R_1{}' = R_1 - R_2 : \quad \Longrightarrow E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$A^{-1} = E_2 E_1$ and $A = (E_2 E_1)^{-1} = E_1^{-1} E_2^{-1}$

**Example:** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

**a)** Find the inverse of $A$.

**b)** Write $A$ and $A^{-1}$ as products of elementary matrices.

**Solution: a)**

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3' = R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -4 & 0 & 1 \end{array}\right] \quad R_2 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \quad R_2' = \frac{1}{2}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$.

Consider the row operations that we had in the previous slide.

- $R_3' = \underbrace{R_3 - 4R_1}_{a_{31}=-4} \Longrightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$
- $R_2 \longleftrightarrow R_3 \Longrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$
- $R_2' = \frac{1}{2}R_2 \Longrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

**b)** $E_3E_2E_1A = I$, and so

$A^{-1} = E_3E_2E_1$ and $A = E_1^{-1}E_2^{-1}E_3^{-1}$.

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$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = A^{-1}$$

**Exercise:** Let

$$A = \begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 2 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Verify that $CA = I_2$. Is $A$ invertible? Explain your answer.

**Example:** Let $A$ be a matrix such that $A^3 = 0$. Use this to simplify $(I-A)(I+A+A^2)$, and then express $(I-A)^{-1}$ in terms of $I$ and $A$.

**Solution:** $(I - A)(I + A + A^2) = I$

$$(I - A)^{-1} = I + A + A^2$$

**Properties of Invertible Matrices**

Let $A$ and $B$ be $n \times n$ invertible matrices.

$$(A^{-1})^{-1} = A,$$

$$(AB)^{-1} = B^{-1}A^{-1},$$

$$(kA)^{-1} = (1/k)A^{-1},$$

$$(A^T)^{-1} = (A^{-1})^T.$$

**Example:**

Let $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$.

Find $(AB)^{-1}$, $(5A)^{-1}$, $(B^T)^{-1}$, and $((AB)^T)^{-1}$.

**Solution:**

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 2 & 4 \end{bmatrix}.$$

$$(5A)^{-1} = \frac{1}{5}A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ -1/5 & 1/5 \end{bmatrix}.$$

$$(B^T)^{-1} = (B^{-1})^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

$$((AB)^T)^{-1} = ((AB)^{-1})^T = \begin{bmatrix} -2 & 5 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 \\ 5 & 4 \end{bmatrix}.$$

**Example:** Find $A$ when $(A^T - 2I)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

**Solution:**

$$\begin{aligned}
(A^T - 2I)^{-1} &= 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}. \\
(A^T - 2I) &= \left(2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \\
&= \frac{1}{2}\left(\frac{1}{3-2}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 3/2 & -1/2 \\ -1 & 1/2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
A^T &= 2I + \begin{bmatrix} 3/2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \\
&= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3/2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 7/2 & -1/2 \\ -1 & 5/2 \end{bmatrix} \Longrightarrow A = \begin{bmatrix} 7/2 & -1 \\ -1/2 & 5/2 \end{bmatrix}.
\end{aligned}$$