MAT1341	**Introduction to Linear Algebra**	**Assignment 2**
section 1341B	prof. José Malagón-López.	due: 30 September

- Hand in your solutions to the following questions at the beginning of class on the indicated date. Late assignments will not be accepted.

- You must explain your reasoning. Unless the question specifically says so, it is understood that all answers must be justified. It is your responsibility to convince the marker that you understand your solution. A correct answer with no reasoning or explanation will not be given any credit.

- All work handed in must be your own. Do not plagiarize.

- Clearly write your name and student number on each page of your solutions.

- Staple all pages of your work together. If you do not do this and some pages are lost, you will not receive credit for your work. It is not the marker's job to track down missing pages.

- Hand in your assignment to the section you are registered in.

[6] 1. Let $\mathbb{M}_2$ denote the set of all matrices of 2×2. Determine if $\mathbb{M}_2$ is a vector space when considered with the standard addition of vectors, but with scalar multiplication given by

$$\alpha \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & b \\ c & \alpha d \end{pmatrix}.$$

In case $\mathbb{M}_2$ fails to be a vector space with these definitions, list at least one axiom that fails to hold. Justify your answer.

Solution: Recall that the standard addition for matrices in $\mathbb{M}_2$ is given by $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$. Already we have seen that $\mathbb{M}_2$ is closed under $+$ and that addition satisfies commutativity, associativity, and existence of additive identity and additive inverse.

Now, let let α and β be two scalars. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be any two matrices. We have that

- $\alpha \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & b \\ c & \alpha d \end{pmatrix} \in \mathbb{M}_2$, so $\mathbb{M}_2$ is closed under $\star$.

- $1 \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- $(\alpha\beta) \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (\alpha\beta)a & b \\ c & (\alpha\beta)d \end{pmatrix} = \alpha \star \begin{pmatrix} \beta a & b \\ c & \beta d \end{pmatrix} = \alpha \star \left(\beta \star \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$.

- $\alpha \star \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \begin{pmatrix} \alpha(a+a') & b+b' \\ c+c' & \alpha(d+d') \end{pmatrix} = \begin{pmatrix} \alpha a + \alpha a' & b+b' \\ c+c' & \alpha d + \alpha d' \end{pmatrix}$
$= \alpha \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \alpha \star \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

- Since

$$(\alpha + \beta) \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)a & b \\ c & (\alpha+\beta)d \end{pmatrix},$$

and

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)a & 2b \\ 2c & (\alpha+\beta)d \end{pmatrix},$$

in general we have that

$$(\alpha + \beta) \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus, $\mathbb{M}_2$ is not a vector space when considered with the standard addition and scalar product given by $\star$ since $(\alpha + \beta) \star A \neq \alpha A + \beta A$, for all $A \in \mathbb{M}_2$.

2. Determine if the following subsets of $\mathbb{R}^3$ are subspaces when considered with the standard operations of addition and scalar multiplication. Justify your answer.

[2] a) $S = \left\{ \begin{pmatrix} x & y & z \end{pmatrix}^T \mid x + y + z = 1 \right\}.$

Solution: For short, notice that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S$ since $0+0+0 \neq 1$. Thus, S is not a subspace of $\mathbb{R}^3$ since the additive identity for the standard addition is not in S.

You also could have shown that S is neither closed under addition or closed under scalar multiplication.

[2] b) $S = \left\{ \begin{pmatrix} x & y & z \end{pmatrix}^T \mid x + z = y \right\}.$

Solution: Notice that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$ since $0 + 0 = 0$, so we must check that S is closed under addition and scalar multiplication.

- If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$ are in S, by definition we have

(1) $$a + c = b \qquad \text{and} \qquad a' + c' = b'$$

  Now, since $\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a + a' \\ b + b' \\ c + c' \end{pmatrix}$, from the identities (1) we obtain that

$$(a + a') + (c + c') = (a + b) + (a' + b') = c + c',$$

  so S is closed under addition.

- If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in S and α is any scalar, we have that $\alpha \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \end{pmatrix}$, so from identity (1) we conclude that $\alpha a + \alpha c = \alpha(a + c) = \alpha b$.

  This implies that S is closed under scalar multiplication.

Thus, S is a subspace of $\mathbb{R}^3$.

[2] c) $S = \left\{ \begin{pmatrix} x & y & z \end{pmatrix}^T \mid 2x^2 + 3y - z = 0 \right\}$

Solution: Notice that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S$ since $2 \cdot 0^2 + 3 \cdot 0 - 0 = 0$, so we must check that S is closed under addition and scalar multiplication.

- If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$ are in S, by definition we have

(2) $$2a^2 + 3b - c = 0 \qquad \text{and} \qquad 2(a')^2 + 3b' - c' = 0.$$

Now, since $\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \\ c+c' \end{pmatrix}$, from the identities (2) we obtain that

$$2(a+a')^2+3(b+b')+(c-c') = 2a^2+4aa'+2(a')^2+3b+3b'-c-c' = (2a^2+3b-c)+(2(a')^2+3b'-c')+4aa' = 4aa'.$$

Since in general $4aa' \neq 0$, S is NOT closed under addition.

Thus, S is not a subspace of $\mathbb{R}^3$.

3. Let $\mathbb{M}_n$ denote the vector space of all the matrices of $n \times n$ with entries in the real numbers. Let $\mathbb{P}$ denote the vector space of all polynomials with coefficients in the real numbers. We are considering both spaces with the standard operations of addition and scalar multiplication. Determine which of the following subsets are subspaces of $\mathbb{M}_n$ or $\mathbb{P}$, according to the case, when considered with the usual operations. Justify your answer.

[3] a) The set of all the matrices $A \in \mathbb{M}_n$ such that $A^T = A$. Such matrices are called *symmetric*.

Solution: Since the transpose of the zero matrix is the zero matrix we have that the zero matrix of size n is symmetric. Given any symmetric matrices (a_{ij}) and (b_{ij}) of $n \times n$, and scalars α and β we have that

$$(\alpha(a_{ij}) + \beta(b_{ij}))^T = (\alpha a_{ij} + \beta b_{ij})^T = (\alpha a_{ji} + \beta b_{ji}) = (\alpha a_{ji}) + (\beta b_{ji}) = \alpha\,(a_{ji}) + \beta\,(b_{ji}) = \alpha\,(a_{ij})^T + \beta\,(b_{ij})^T.$$

Thus, the set of all the $n \times n$ symmetric matrices is a subspace of $\mathbb{M}_n$.

[3] b) The set $S = \{P(x) \in \mathbb{P} \mid P(1) = 0\}$.

Solution: Clearly the zero polynomial is in S. Now, given $P(x)$ and $P'(x)$ in S, and scalars α and β, we have that

$$(\alpha P(x) + \beta P'(x))\,(1) = \alpha P(1) + \beta P'(1) = \alpha 0 + \beta 0 = 0.$$

Thus, S is a subspace of $\mathbb{P}$.

[3] c) The set of all the matrices $(a_{ij}) \in \mathbb{M}_n$ such that $a_{ij} = 0$ for all $i > j$. In other words, all the matrices whose entries lying below the diagonal are zero. Such matrices are called *upper triangular*.

Solution: Notice that the zero matrix is an upper triangular. Now, given any upper triangular matrices (a_{ij}) and (b_{ij}) of $n \times n$, and scalars α and β we have that

$$\alpha(a_{ij}) + \beta(b_{ij}) = (\alpha a_{ij} + \beta b_{ij})\,.$$

Since (a_{ij}) and (b_{ij}) are upper triangular, $a_{ij} = b_{ij} = 0$ for all $i > j$, so $\alpha a_{ij} + \beta b_{ij} = 0$ for all $i > j$. Thus, the set of all the $n \times n$ upper triangular matrices is a subspace of $\mathbb{M}_n$.

[**/21**]