MATH 3705* Test 4 - SOLUTIONS March 2008

1. [6 marks] Find the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangle $0 < x < 2,\ 0 < y < 1$, which satisfies the boundary conditions $u(0, y) = 2y,\ u(2, y) = 0$, $u(x, 0) = 0,\ u(x, 1) = 2 - x$. Write down the complete solution $u(x, y)$.

Solution:

The boundary condition is continuous, and linear on each portion of the boundary, so

$$u(x, y) = \alpha x + \beta y + \gamma xy + \delta.$$

(1) The lower horizontal segment: $y = 0 \Rightarrow \alpha x + \delta = 0 \Rightarrow \alpha = \delta = 0$
$\Rightarrow u(x, y) = \beta y + \gamma xy.$

(2) The left vertical segment: $x = 0 \Rightarrow \beta y = 2y \Rightarrow \beta = 2$
$\Rightarrow u(x, y) = 2y + \gamma xy.$

(3) The right vertical segment: $x = 2 \Rightarrow 2y + 2\gamma y = 0 \Rightarrow \gamma = -1$
$\Rightarrow u(x, y) = 2y - xy.$

Check the upper horizontal segment: $y = 1 \Rightarrow 2 - x = 2 - x$. Thus,

$$u(x, y) = 2y - xy.$$

2. [6] The bounded solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = a$ has the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

Find the bounded solution of Laplace's equation outside the circle $r = 2$, subject to the boundary condition $u(2, \theta) = 1 - \cos(\theta) + \sin(2\theta)$.

Solution:

$$1 - \cos(\theta) + \sin(2\theta) = u(2, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

$\Rightarrow \frac{a_0}{2} = 1,\ a_1 = -2,\ b_2 = 4$, and $a_n = b_n = 0$ otherwise. Thus,

$$u(r, \theta) = 1 - 2r^{-1}\cos(\theta) + 4r^{-2}\sin(2\theta)$$

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3. [4] The solution of the wave equation $u_{xx} = \dfrac{1}{9}u_{tt},\ 0 < x < 2$, which satisfies the boundary conditions $u(0,t) = u(2,t) = 0$, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left\{a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right)\right\}.$$

If $u(x,t)$ satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 3\sin(\pi x) - \sin(3\pi x)$, find the coefficients a_n and b_n.

Solution:

$b_2 = \dfrac{1}{\pi},\ b_6 = -\dfrac{1}{9\pi},\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

4. [14] The solution of the heat equation $w_{xx} = \dfrac{1}{\alpha^2}w_t,\ 0 < x < L$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\dfrac{\alpha^2 n^2 \pi^2}{L^2}t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \dfrac{1}{9}u_t,\ 0 < x < 3$, which satisfies the boundary conditions $u(0,t) = -1,\ u(3,t) = 2$, and the initial condition $u(x,0) = x$. Write down the complete solution $u(x,t)$ and the first four terms.

Solution:

$L = 3,\ \alpha = 3.$ The boundary conditions are nonhomogeneous, therefore

$$u(x,t) = v(x) + w(x,t),$$

where $w(x,t)$ satisfies the PDE with the homogeneous boundary conditions, and $v(x)$ satisfies

$$v''(x) = 0, \quad v(0) = -1, \quad v(3) = 2.$$

Thus, $v(x) = x - 1$, and $w(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\dfrac{\alpha^2 n^2 \pi^2}{L^2}t} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t}.$

It remains to find b_n, which we do by satisfying the initial condition

$$u(x,0) = v(x) + w(x,0) = x.$$

It follows that

$$w(x,0) = u(x,0) - v(x) = x - (x-1) = 1,$$

so b_n is the Fouries sine coefficient of 1, and therefore

$$b_n = \frac{2}{3}\int_0^3 \sin(\frac{n\pi x}{3})\,dx = \frac{-2}{n\pi}\cos(\frac{n\pi x}{3})\Big|_0^3 = \frac{2}{n\pi}[1-(-1)^n].$$

Finally,

$$u(x,t) = v(x) + w(x,t) = x - 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t} =$$

$$= x - 1 + \frac{4}{\pi}\sin(\frac{\pi x}{3})e^{-\pi^2 t} + \frac{4}{3\pi}\sin(\pi x)e^{-9\pi^2 t} - ...$$