MATH 3705* Test 4 - SOLUTIONS March 2008

1. [4] Consider the Sturm-Liouville differential equation $2x^2y'' - 3xy' + 2\lambda y = 0$. Find the weight function $r(x)$.

Solution:

The equation in standard for is $y'' - \dfrac{3}{2x}y' + \dfrac{1}{x^2}\lambda y = 0$.
The integrating factor is $\mu = e^{-\frac{3}{2}\int \frac{1}{x}\,dx} = e^{-\frac{3}{2}\ln x} = x^{-\frac{3}{2}}$.
Then the equation in the S-L form is $(x^{-3/2}y')' + \dfrac{x^{-3/2}}{x^2}\lambda y = 0$, so $r(x) = \dfrac{x^{-3/2}}{x^2} = x^{-7/2}$.

2. [6] The bounded solution of Laplace's equation $u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = a$ has the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right].$$

Find the bounded solution of Laplace's equation outside the circle $r = 2$, subject to the boundary condition $u(2,\theta) = 1 - \cos(\theta) + \sin(2\theta)$.

Solution:

$1 - \cos(\theta) + \sin(2\theta) = u(2,\theta) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right]$
$\Rightarrow \dfrac{a_0}{2} = 1,\ a_1 = 2,\ b_2 = 4$, and $a_n = b_n = 0$ otherwise. Thus,

$$u(r,\theta) = 1 + 2r^{-1}\cos(\theta) + 4r^{-2}\sin(2\theta)$$

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3. [20 marks] Find all eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem

$$y'' + 2y' + \lambda y = 0,\ y(0) = 0,\ y(1) = 0$$

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Solution:

$y = e^{\alpha x} \ \Rightarrow\ \alpha^2 + 2\alpha + \lambda = 0 \ \Rightarrow\ \alpha = \dfrac{-2 \pm \sqrt{4 - 4\lambda}}{2} = -1 \pm \sqrt{1 - \lambda}.$

$\lambda > \mathbf{1}$:
$\Rightarrow\ \alpha = -1 \pm i\sqrt{\lambda - 1} \Rightarrow\ y = e^{-x}[A\cos(\sqrt{\lambda - 1}\ x) + B\sin(\sqrt{\lambda - 1}\ x)].$
$y(0) = 0 \ \Rightarrow\ A = 0 \ \Rightarrow\ y = Be^{-x}\sin(\sqrt{\lambda - 1}\ x).$
$y(1) = 0 \ \Rightarrow\ Be^{-1}\sin(\sqrt{\lambda - 1}) = 0 \ \Rightarrow\ \sqrt{\lambda - 1} = n\pi.$
Thus, $\lambda_n = n^2\pi^2 + 1,\ n \geq 1$, are the eigenvalues, and $y_n = B_n e^{-x}\sin(n\pi x)$ are the corresponding eigenfunctions.

$\lambda = \mathbf{1}$:
$\Rightarrow\ \alpha = -1 \Rightarrow\ y = e^{-x}[A + Bx].$
$y(0) = 0 \ \Rightarrow\ A = 0 \ \Rightarrow\ y = Bxe^{-x}.$
$y(1) = 0 \ \Rightarrow\ Be^{-1} = 0 \ \Rightarrow\ B = 0 \Rightarrow\ y = 0.$

$\lambda < \mathbf{1}$:
$\Rightarrow\ \alpha = -1 \pm \sqrt{1 - \lambda} \Rightarrow\ y = Ae^{(-1+\sqrt{1-\lambda})x} + Be^{(-1-\sqrt{1-\lambda})x}.$
$y(0) = y(1) = 0 \ \Rightarrow\ \begin{cases} \mathrm{A} + \mathrm{B} = 0 \\ \mathrm{A}e^{(-1+\sqrt{1-\lambda})} + \mathrm{B}e^{(-1+\sqrt{1-\lambda})} = 0 \end{cases} \Rightarrow\ A = B = 0$, since $\begin{vmatrix} 1 & 1 \\ e^{(-1+\sqrt{1-\lambda})} & e^{(-1+\sqrt{1-\lambda})} \end{vmatrix} \neq 0.$