MATH 3705* B   Test 3   Answers and solutions   March 2009

Questions 1-5 are multiple choice.

1. [3] The general solution of $xy'' + y' + 7xy = 0$ for $x > 0$ is

   (a) $c_1 J_0(\sqrt{7}\,x) + c_2 J_0(\sqrt{7}\,x)\ln(x)$ (b) $c_1 J_0(\sqrt{7}\,x) + c_2 Y_0(\sqrt{7}\,x)$
   (c) $c_1 J_{\sqrt{7}}(x) + c_2 J_{-\sqrt{7}}(x)$ (d) $c_1 J_{\sqrt{7}}(x) + c_2 Y_{\sqrt{7}}(x)$ (e) None of these

2. [3] The general solution of $x^2y'' + xy' + (5x^2 - 9)y = 0$ for $x > 0$ is

   (a) $c_1 J_3(\sqrt{5}\,x) + c_2 J_{-3}(\sqrt{5}\,x)$ (b) $c_1 J_3(\sqrt{5}\,x) + c_2 Y_3(\sqrt{5}\,x)$
   (c) $c_1 J_{\sqrt{5}}(3x) + c_2 J_{-\sqrt{5}}(3x)$ (d) $c_1 J_{\sqrt{5}}(3x) + c_2 Y_{\sqrt{5}}(3x)$ (e) None of these

3. [2] At $x = 17$, the Fourier cosine series of $f(x) = \left\{ \begin{array}{l} 1,\ 0 < x < 2 \\ 0,\ 2 < x < 3 \end{array} \right\}$ converges to

   (a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1 (e) None of these

4. [2] The function $x^2 \sin(x) + x^3 \cos(x)$ is

   (a) even; (b) odd; (c) neither .

5. [2] If f is a periodic function of period 5, and $f(x) = x^3,\ 0 < x < 5$, then $f(100)$ is

   (a) 0; (b) 1; (c) 8; (d) 64; (e) 125.

Answers: b, b, d, b, a.

6. [8 marks] Find the Fourier cosine series of $f(x) = x - 3$ on $[0, \pi]$. Give the first three terms of the series.

Solution:

$a_0 = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} (x-3)\, dx = \pi - 6.$

For $n \geq 1$, $a_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} (x-3)\, \cos(nx)\, dx = \dfrac{2}{\pi} \left\{ \dfrac{x-3}{n} \sin(nx) + \dfrac{1}{n^2} \cos(nx) \right\}_0^{\pi} =$

$= \dfrac{2}{n^2\pi}[\cos(n\pi) - cos(0)] = \dfrac{2}{n^2\pi}[(-1)^n - 1].$

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Then the series is

$$\frac{\pi-6}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\Big((-1)^n - 1\Big) \cos(nx) = \frac{\pi}{2} - 3 - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - ...$$

7. [10 marks] The solution of the heat equation $w_{xx} = \dfrac{1}{\alpha^2} w_t,\ 0 < x < L$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\dfrac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \dfrac{1}{9} u_t,\ 0 < x < 3$, which satisfies the boundary conditions $u(0,t) = -1,\ u(3,t) = 2$, and the initial condition $u(x,0) = x$. Write down the complete solution $u(x,t)$ (give the first three terms).

Solution:

$L = 3,\ \alpha = 3.$ The boundary conditions are homogeneous, therefore it remains to find b_n as the Fouries sine coefficient of $f(x) = 1 - 3x$ on the interval $[0,3]$:

$$b_n = \frac{2}{3} \int_0^3 (1-3x)\, \sin(\frac{n\pi x}{3})\, dx = \frac{2}{3} \left\{ \frac{3(3x-1)}{n\pi} \cos(\frac{n\pi x}{3}) + \frac{27}{n^2\pi^2} \sin(\frac{n\pi x}{3}) \right\}_0^3 =$$

$$= \frac{2}{n\pi} \left\{ 8\cos(n\pi) + \frac{9}{n\pi} \sin(n\pi) - (-\cos(0)) - \frac{9}{n\pi} \sin(0) \right\} = \frac{2}{n\pi} \left\{ 8(-1)^n + 1 \right\}.$$

Thus,

$$b_n = \begin{cases} \dfrac{18}{n\pi}, & n \text{ even}, \\[2ex] \dfrac{-14}{n\pi}, & n \text{ odd} \end{cases}$$

Finally,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t} =$$

$$= \frac{-14}{\pi} \sin(\frac{\pi x}{3}) e^{-\pi^2 t} + \frac{9}{\pi} \sin(\frac{2\pi x}{3}) e^{-4\pi^2 t} - \frac{6}{\pi} \sin(\pi x) e^{-9\pi^2 t} + \ldots$$