MATH 3705* Test 3 March 2008

LAST NAME: ______________________ **ID#:** ______________

Questions 1-3 are multiple choice. Circle the correct answer. Only the answer will be marked.

1. [2 marks] At $x = 121$, the Fourier **cosine** series of the 6-periodic function

$$f(x) = \left\{ \begin{array}{l} 1,\ 0 \square x < 2 \\ 0,\ 2 \square x < 3 \end{array} \right\}$$

converges to

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $-\frac{1}{2}$ (e) None of these

2. [2 marks] At $x = 20$, the Fourier **sine** series of the 4-periodic function

$$f(x) = \left\{ \begin{array}{l} 1,\ 0 \square x < 1 \\ 0,\ 1 \square x < 2 \end{array} \right\}$$

converges to

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1 (e) None of these

3. [3 marks] The solution of the wave equation $u_{xx} = \frac{1}{9}u_{tt},\ 0 < x < 2$, which satisfies the boundary conditions $u(0,t) = u(2,t) = 0$, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \square a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right)\Big] .$$

If $u(x,t)$ satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 6\sin(\pi x) + \sin(3\pi x)$, the coefficients a_n and b_n are given by

(a) $a_2 = 6,\ a_6 = 1,\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(b) $a_2 = \frac{2}{\pi},\ a_6 = \frac{1}{9\pi},\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(c) $b_2 = 6,\ b_6 = 1,\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(d) $b_2 = \frac{2}{\pi},\ b_6 = \frac{1}{9\pi},\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(e) None of the above

Answers: c, a, d.

4. [5 marks] Find the Fourier cosine series of $f(x) = 1 - x$ on $[0, \pi]$. Give the first three terms of the series.

Solution: $L = \pi$.

$$a_0 = \frac{2}{\pi}\int_0^\pi (1-x)\,dx = \frac{2}{\pi}\left\{x - \frac{x^2}{2}\right\}_0^\pi = 2 - \pi.$$

For $n \geq 1$, $a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi (1-x)\,\cos(nx)\,dx = \frac{2}{\pi}\left\{\frac{1-x}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right\}_0^\pi =$

$$= \frac{2}{n^2\pi}[\cos(0) - \cos(n\pi)] = \frac{2}{n^2\pi}[1 - (-1)^n].$$

Then the series is

$$\frac{2-\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[1 - (-1)^n]\cos(nx) = 1 - \frac{\pi}{2} + \frac{4}{\pi}\cos(x) + \frac{4}{9\pi}\cos(3x) + \ldots$$

5. [9 marks] The solution of the heat equation $w_{xx} = \dfrac{1}{\alpha^2} w_t,\ 0 < x < L$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\dfrac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \dfrac{1}{9} u_t,\ 0 < x < 3$, which satisfies the boundary conditions $u(0,t) = -1,\ u(3,t) = 2$, and the initial condition $u(x,0) = x$. Write down the complete solution $u(x,t)$ (give the first three terms).

Solution:

$L = 3,\ \alpha = 3$. The boundary conditions are nonhomogeneous, therefore

$$u(x,t) = v(x) + w(x,t),$$

where $w(x,t)$ satisfies the PDE with the homogeneous boundary conditions, and $v(x)$ satisfies

$$v''(x) = 0, \quad v(0) = -1, \quad v(3) = 2.$$

Thus, $v(x) = x - 1$, and $w(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\dfrac{\alpha^2 n^2 \pi^2}{L^2} t} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t}$.

It remains to find b_n, which we do by satisfying the initial condition

$$u(x,0) = v(x) + w(x,0) = x.$$

It follows that

$$w(x,0) = u(x,0) - v(x) = x - (x-1) = 1,$$

so b_n is the Fouries sine coefficient of 1, and therefore

$$b_n = \frac{2}{3}\int_0^3 \sin\left(\frac{n\pi x}{3}\right) dx = \frac{2}{n\pi} \cos\left(\frac{n\pi x}{3}\right)\Big|_0^3 = \frac{2}{n\pi}[(-1)^n - 1].$$

Finally,

$$u(x,t) = v(x) + w(x,t) = x - 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}[(-1)^n - 1] \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t} =$$

$$= x - 1 - \frac{4}{\pi} \sin\left(\frac{\pi x}{3}\right) e^{-\pi^2 t} - \frac{4}{3\pi} \sin(\pi x) e^{-3\pi^2 t} - \ldots$$

6. [9 marks] The solution of the wave equation $w_{xx} = \frac{1}{c^2} w_{tt},\ 0 < x < L, t > 0$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) \left\{ a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}) \right\}.$$

Find the solution $u(x,t)$ of $u_{xx} = 4u_{tt},\ 0 < x < \pi$, which satisfies the boundary conditions $u(0,t) = u(\pi,t) = 0$, and the initial condition $u(x,0) = f(x) = 0,\ \ u_t(x,0) = g(x) = x$. Write down the complete solution $u(x,t)$.

Solution:

$L = \pi,\ c = \frac{1}{2},$ and therefore

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx) \left\{ a_n \cos\left(\frac{nt}{2}\right) + b_n \sin\left(\frac{nt}{2}\right) \right\} \qquad (*)$$

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The first initial condition $u(x,0) = 0$ implies that $\sum_{n=1}^{\infty} a_n \sin(nx) = 0$ for all x, and therefore $a_n = 0$ for all $n \geq 1$.

To satisfy the second initial condition $u_t(x,0) = x$, we differentiate $u(x,t)$ given by $(*)$ with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(nx) \left\{ -\frac{n}{2}\, a_n \sin\left(\frac{nt}{2}\right) + \frac{n}{2}\, b_n \cos\left(\frac{nt}{2}\right) \right\}.$$

Then

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n}{2}\, b_n \sin(nx) = g(x) = x,$$

where $\frac{n}{2} b_n$ is the Fourier sine coefficient of $g(x) = x$ on the interval $[0,\pi]$:

$$\frac{n}{2}\, b_n = \frac{2}{\pi} \int_0^{\pi} x\, \sin(nx)\, dx = \frac{2}{\pi} \left\{ -\frac{x}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) \right\}_0^{\pi} =$$
$$= \frac{2}{\pi} \left\{ -\frac{\pi}{n} \cos(n\pi) \right\} = \frac{2}{n}(-1)^{n+1} \qquad \Rightarrow \qquad b_n = \frac{2}{n} \cdot \frac{2}{n}(-1)^{n+1} = \frac{4}{n^2}(-1)^{n+1}.$$

Thus, the complete solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^{n+1}\, \sin(nx)\, \sin\left(\frac{nt}{2}\right).$$