MATH 3705* B **Test 3** **Answers and solutions** March 2007

Questions 1-3 are multiple choice. Circle the correct answer. Only the answer will be marked.

1. [3 marks] The general solution of $xy'' + y' + 7xy = 0$ for $x > 0$ is

$(a)\ c_1 J_0(\sqrt{7}\,x) + c_2 J_0(\sqrt{7}\,x)\ln(x)$ $(b)\ c_1 J_0(\sqrt{7}\,x) + c_2 Y_0(\sqrt{7}\,x)$

$(c)\ c_1 J_{\sqrt{7}}(x) + c_2 J_{-\sqrt{7}}(x)$ $(d)\ c_1 J_{\sqrt{7}}(x) + c_2 Y_{\sqrt{7}}(x)$ (e) None of the above

2. [2 marks] At $x = 17$, the Fourier cosine series of $f(x) = \left\{ \begin{array}{l} 1,\ 0 \,\Box\, x < 2 \\ 0,\ 2 \,\Box\, x < 3 \end{array} \right\}$ converges to

(a) 0 (b) $\dfrac{1}{2}$ (c) $-\dfrac{1}{2}$ (d) 1 (e) None of these

3. [3 marks] The solution of the wave equation $u_{xx} = \dfrac{1}{9} u_{tt}$, $0 < x < 2$, which satisfies the boundary conditions $u(0,t) = u(2,t) = 0$, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \left\{ a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right) \right\}.$$

If $u(x,t)$ satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 3\sin(\pi x) - \sin(3\pi x)$, the coefficients a_n and b_n are given by

(a) $b_2 = 3,\ b_6 = -1,\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(b) $b_2 = \dfrac{1}{\pi},\ b_6 = -\dfrac{1}{9\pi},\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(c) $a_2 = -3,\ a_6 = 1,\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(d) $a_2 = -\dfrac{1}{\pi},\ a_6 = \dfrac{1}{9\pi},\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(e) None of the above

Answers: **b, d, b.**

4. [6 marks] Find the Fourier cosine series of $f(x) = x - 3$ on $[0, \pi]$. Give the first three terms of the series.

Solution:

$a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi (x-3)\,dx = \pi - 6.$

For $n \geq 1$, $\;a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi (x-3)\,\cos(nx)\,dx = \dfrac{2}{\pi}\left\{-\dfrac{x-3}{n}\sin(nx) + \dfrac{1}{n^2}\cos(nx)\right\}_0^\pi =$

$= \dfrac{2}{n^2\pi}[\cos(n\pi) - cos(0)] = \dfrac{2}{n^2\pi}[(-1)^n - 1].$

Then the series is

$$\frac{\pi - 6}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\Big((-1)^n - 1\Big)\cos(nx) = \frac{\pi}{2} - 3 - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - \ldots$$

5. [8 marks] The solution of the heat equation $w_{xx} = \frac{1}{\alpha^2} w_t,\ 0 < x < L$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = 9u_t,\ 0 < x < 4$, which satisfies the boundary conditions $u(0,t) = -1,\ u(4,t) = 3$, and the initial condition $u(x,0) = x + 1$. Write down the complete solution $u(x,t)$.

Solution:

$L = 4,\ \alpha = \frac{1}{3}$. The boundary conditions are nonhomogeneous, therefore

$$u(x,t) = v(x) + w(x,t),$$

where $w(x,t)$ satisfies the PDE with the homogeneous boundary conditions, and $v(x)$ satisfies

$$v''(x) = 0, \quad v(0) = -1, \quad v(4) = 3.$$

Thus, $v(x) = x - 1$, and $w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right) e^{-\frac{n^2 \pi^2}{144} t}$.

It remains to find b_n, which we do by satisfying the initial condition

$$u(x,0) = v(x) + w(x,0) = x + 1.$$

It follows that

$$w(x,0) = u(x,0) - v(x) = x + 1 - (x - 1) = 2,$$

so b_n is the Fouries sine coefficient of 2, and therefore

$$b_n = \frac{2}{4} \int_0^4 2 \sin(\frac{n\pi x}{4})\, dx = -\frac{4}{n\pi} \cos(\frac{n\pi x}{4})\Big|_0^4 = \frac{4}{n\pi}[1 - (-1)^n].$$

Finally,

$$u(x,t) = v(x) + w(x,t) = x + 1 + \sum_{n=1}^{\infty} \frac{4}{n\pi}[1 - (-1)^n] \sin\left(\frac{n\pi x}{4}\right) e^{-\frac{n^2\pi^2}{144} t} =$$

$$= x + 1 + \frac{8}{\pi} \sin(\frac{\pi x}{4}) e^{-\frac{\pi^2}{144} t} + \frac{8}{3\pi} \sin(\frac{3\pi x}{4}) e^{-\frac{\pi^2}{48} t} + ...$$

6. [8 marks] The solution of the wave equation $w_{xx} = \frac{1}{c^2} w_{tt},\ 0 < x < L, t > 0$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) \left\{ a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}) \right\}.$$

Find the solution $u(x,t)$ of $u_{xx} = \frac{1}{9} u_{tt},\ 0 < x < \pi$, which satisfies the boundary conditions $u(0,t) = u(\pi,t) = 0$, and the initial condition $u(x,0) = f(x) = 0,\ \ u_t(x,0) = g(x) = \pi - x$. Write down the complete solution $u(x,t)$.

Solution:

$L = \pi,\ c = 3,$ and therefore

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx) \left\{ a_n \cos(3nt) + b_n \sin(3nt) \right\}$$

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The first initial condition $u(x,0) = 0$ implies that $\sum_{n=1}^{\infty} a_n \sin(nx) = 0$ for all x, and therefore $a_n = 0$ for all $n \geq 1$.

To satisfy the second initial condition $u_t(x,0) = \pi - x$, we calculate

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(nx) \left\{ -3n\, a_n \sin(3nt) + 3n\, b_n \cos(3nt) \right\}.$$

Then

$$u_t(x,0) = \sum_{n=1}^{\infty} 3n\, b_n \sin(nx) = \pi - x,$$

where $3nb_n$ is the Fourier sine coefficient of $g(x) = \pi - x$ on the interval $[0, \pi]$:

$$3n\, b_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin(nx)\, dx = \frac{2}{\pi} \left\{ (\pi - x)(-\frac{1}{n} \cos(nx)) - \frac{1}{n^2} \sin(nx) \right\}_0^{\pi} =$$
$$= \frac{2}{\pi} \left\{ \frac{\pi}{n} \cos(0) \right\} = \frac{2}{n} \qquad \Rightarrow \qquad b_n = \frac{2}{n} \cdot \frac{1}{3n} = \frac{2}{3n^2}.$$

Thus, the complete solution is

$$u(x,t) = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)\, \sin(3nt).$$