MATH 3705A
Test 3
March 2006

LAST NAME: ____________________________ **ID#:**______________

Questions 1 and 2 are multiple choice, worth 5 marks each. Circle the correct answer. Only the answer will be marked.

1. At $x = 27$, the Fourier sine series of $f(x) = \left\{ \begin{array}{l} 1,\ 0 \le x < 1 \\ 0,\ 1 \le x \le 2 \end{array} \right\}$ converges to

(a) 0 (b) $-\dfrac{1}{2}$ (c) $\dfrac{1}{2}$ (d) 1 (e) None of these

2. The solution of the wave equation $u_{xx} = \frac{1}{9}u_{tt},\ 0 < x < 2$, which satisfies the boundary conditions $u(0,t) = u(2,t) = 0$, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \left[a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right) \right].$$

If $u(x,t)$ satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 3\sin(\pi x) - \sin(3\pi x)$, the coefficients a_n and b_n are given by

(a) $b_2 = \dfrac{1}{\pi},\ b_6 = -\dfrac{1}{9\pi},\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(b) $a_2 = -3,\ a_6 = 1,\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(c) $b_2 = 3,\ b_6 = -1,\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(d) $a_2 = -\dfrac{1}{\pi},\ a_6 = \dfrac{1}{9\pi},\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(e) None of the above

Answers: **b, a.**

3. [6 marks] Find the Fourier cosine series of $f(x) = x$ on $[0, \pi]$.

Solution:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi, \text{ and for } n \geq 1, \; a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{n\pi} x\sin(nx)\Big|_0^{\pi} -$$

$$\frac{2}{n\pi}\int_0^{\pi} \sin(nx)\,dx = \frac{2}{n^2\pi}\cos(nx)\Big|_0^{\pi} = \frac{2}{n^2\pi}\square(-1)^n - 1\Big].$$

Then the series is

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\square(-1)^n - 1\Big]\cos(nx) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - \ldots$$

4. [8 marks] The solution of the heat equation $w_{xx} = \dfrac{1}{\alpha^2} w_t,\ 0 < x < L$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\dfrac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \dfrac{1}{9} u_t,\ 0 < x < 3$, which satisfies the boundary conditions $u(0,t) = -1,\ u(3,t) = 2$, and the initial condition $u(x,0) = x$. Write down the complete solution $u(x,t)$.

Solution:

$L = 3,\ \alpha = 3$. The boundary conditions are nonhomogeneous, therefore

$$u(x,t) = v(x) + w(x,t),$$

where $w(x,t)$ satisfies the PDE with the homogeneous boundary conditions, and $v(x)$ satisfies

$$v''(x) = 0, \quad v(0) = -1, \quad v(3) = 2.$$

Thus, $v(x) = x - 1$, and $w(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\dfrac{\alpha^2 n^2 \pi^2}{L^2} t} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t}$.

It remains to find b_n, which we do by satisfying the initial condition

$$u(x,0) = v(x) + w(x,0) = x.$$

It follows that

$$w(x,0) = u(x,0) - v(x) = x - (x-1) = 1,$$

so b_n is the Fouries sine coefficient of 1, and therefore

$$b_n = \frac{2}{3}\int_0^3 \sin\left(\frac{n\pi x}{3}\right) dx = \frac{2}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Big|_0^3 = \frac{2}{n\pi}[(-1)^n - 1].$$

Finally,

$$u(x,t) = v(x) + w(x,t) = x - 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}[(-1)^n - 1] \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t} =$$

$$= x - 1 - \frac{4}{\pi}\sin\left(\frac{\pi x}{3}\right)e^{-\pi^2 t} - \frac{4}{3\pi}\sin(\pi x)e^{-3\pi^2 t} - \ldots$$

5. [6 marks] Find the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangle $0 < x < 2,\ 0 < y < 1$, which satisfies the boundary conditions $u(0, y) = 2y,\ u(2, y) = 0,$ $u(x, 0) = 0,\ u(x, 1) = 2 - x$. Write down the complete solution $u(x, y)$.

Solution:

The boundary condition is continuous, and linear on each portion of the boundary, so

$$u(x, y) = \alpha x + \beta y + \gamma xy + \delta.$$

(1) The lower horizontal segment: $y = 0 \ \Rightarrow\ \alpha x + \delta = 0 \ \Rightarrow\ \alpha = \delta = 0$
$\Rightarrow\ u(x, y) = \beta y + \gamma xy.$

(2) The left vertical segment: $x = 0 \ \Rightarrow\ \beta y = 2y \ \Rightarrow\ \beta = 2$
$\Rightarrow\ u(x, y) = 2y + \gamma xy.$

(3) The right vertical segment: $x = 2 \ \Rightarrow\ 2y + 2\gamma y = 0 \ \Rightarrow\ \gamma = -1$
$\Rightarrow\ u(x, y) = 2y - xy.$

Check the upper horizontal segment: $y = 1 \ \Rightarrow\ 2 - x = 2 - x.$

Answer: $u(x, y) = 2y - xy.$