MATH 3705 Test 2 - Answers and Solutions February 2008

LAST NAME: ______________________________ **ID#:** ______________

Questions 1-7 are multiple choice. Circle the correct answer. Only the answer will be marked.

1. [**3 marks**] The general solution of $x^2y'' + 2xy' + y = 0$ for $x \neq 0$ is

(a) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{\frac{\sqrt{3}}{2}}$ (b) $c_1|x|^{-\frac{1}{2}+\frac{\sqrt{3}}{2}} + c_2|x|^{-\frac{1}{2}-\frac{\sqrt{3}}{2}}$ (c) $|x|^{-\frac{1}{2}+\frac{\sqrt{3}}{2}}\left[c_1 + c_2 \ln|x|\right]$

(d) $|x|^{-\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right]$ (e) None of the above

Solution: This is an Euler equation with $A = 2$, $B = 1$. The indicial equation is $r^2 + r + 1 = 0$ with $r_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ $\Rightarrow$ Euler Equation, case (iii) $\Rightarrow$ (d).

2. [**3 marks**] The general solution of $4x^2y'' + 8xy' + y = 0$ for $x \neq 0$ is

(a) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$ (b) $|x|^{-\frac{1}{2}}\left[c_1 + c_2 \ln|x|\right]$ (c) $|x|^{-\frac{1}{2}}\left[c_1 \cos(\ln|x|) + c_2 \sin(\ln|x|)\right]$

(d) $|x|^{-\frac{1}{2}} \left. c_1 \cos\left(-\frac{1}{2}\ln|x|\right) + c_2 \sin\left(-\frac{1}{2}\ln|x|\right)\right]$ (e) None of the above

Solution: This is an Euler equation with $A = \frac{8}{4} = 2$, $B = \frac{1}{4}$. The indicial equation is $r^2 + (2-1)r + \frac{1}{4} = 0$, or $4r^2 + 4r + 1 = (2r+1)^2 = 0$. The roots are $r_1 = r_2 = -\frac{1}{2}$ $\Rightarrow$ Euler Equation, case (ii) $\Rightarrow$ (b).

3. [**2 marks**] The general solution of $x^2y'' + 4xy' - 10y = 0$ for $x \neq 0$ is

(a) $C_1|x|^2 + C_2|x|^{-5}$ (b) $C_1|x|^{-2} + C_2|x|^5$ (c) $|x|^{-2}(C_1 + C_2 \ln|5x|)$

(d) $|x|^{-2}\left[C_1 \cos(5\ln|x|) + C_2 \sin(5\ln|x|)\right]$ (e) None of the above

Solution: The indicial equation is $r(r-1) + 4r - 10 = 0$, or $r^2 + 3r - 10 = 0$, with $r_1 = 2$, $r_2 = -5$, $\Rightarrow$ Euler equation, case (i) $\Rightarrow$ (a).

4. [2 marks] The general solution of $x^2y'' + xy' + (7x^2 - 1)y = 0$ for $x > 0$ is

(a) $c_1 J_1(\sqrt{7}\,x) + c_2 Y_1(\sqrt{7}\,x)$ (b) $c_1 J_1(\sqrt{7}\,x) + c_2 J_{-1}(\sqrt{7}\,x)$
(c) $c_1 J_{\sqrt{7}}(x) + c_2 J_{-\sqrt{7}}(x)$ (d) $c_1 J_{\sqrt{7}}(x) + c_2 Y_{\sqrt{7}}(x)$ (e) None of the above

Solution: This is Bessel's equation of order $\nu = 1$ with parameter $\lambda = \sqrt{7}$. Since ν is an integer, then two linearly independent solutions are given by $J_1(\sqrt{7}\,x)$ and $Y_1(\sqrt{7}\,x) \Rightarrow (a)$.

5. [2 marks] The general solution of $x^2y'' + xy' + 3x^2y = 0$ for $x > 0$ is

(a) $c_1 J_0(\sqrt{3}\,x) + c_2 J_0(\sqrt{3}\,x)\ln(x)$ (b) $c_1 J_0(\sqrt{3}\,x) + c_2 Y_0(\sqrt{3}\,x)$
(c) $c_1 J_{\sqrt{3}}(x) + c_2 J_{-\sqrt{3}}(x)$ (d) $c_1 J_{\sqrt{3}}(x) + c_2 Y_{\sqrt{3}}(x)$ (e) None of the above

Solution: This is Bessel's equation of order $\nu = 0$ with parameter $\lambda = \sqrt{3}$. Two linearly independent solutions are given by $J_0(\sqrt{3}\,x)$ and $Y_0(\sqrt{3}\,x) \Rightarrow (b)$.

6. [2 marks] The equation $xy'' + x^2y' + \dfrac{1}{2(x-1)}\,y = 0$ has

(a) one regular singular point $x = 1$.

(b) one regular singular point $x = 0$.

(c) two regular singular points $x = 0$ and $x = 1$.

(d) one regular singular points $x = 0$ and one irregular singular point $x = 1$.

(e) no singular points.

Solution: The equation in standard form is $y'' + xy' + \dfrac{1}{2x(x-1)}\,y = 0$, $\Rightarrow$ $q(x) = \dfrac{1}{2x(x-1)}$ is not analytic at $x = 0$ and at $x = 1$. However, $x^2q(x) = \dfrac{x}{2(x-1)}$ is analytic at $x = 0$, and $(x-1)^2q(x) = \dfrac{x-1}{2x}$ is analytic at $x = 1$, $\Rightarrow$ (c).

7. [2 marks] The differential equation $2y'' - \dfrac{5}{x-3}y' + 7y = 0$ has a singular point $x_0 = 3$. Then the series solution $y = \displaystyle\sum_{n=0}^{\infty} a_n(x+1)^n$ about $x = -1$ has the radius of convergence

(a) $R = \infty$ (b) $R \geq 4$ (c) $R \geq 3$ (d) $R \geq 1$

Solution: The equation has a singular point $x_0 = 3$, and all other points are ordinary. The distance from the expansion point $x = -1$ to the singular point $x_0 = 3$ is 4 units $\Rightarrow$ (b).

8. **[14 marks]** The differential equation $2x^2y'' - xy' + (1-x)y = 0$ has a regular singular point $x_0 = 0$.

(a) [5] Show that $r_1 = 1$ and $r_2 = \dfrac{1}{2}$ are the roots of the indicial equation.
(b) [8] Find a power series solution, valid for $x > 0$, which corresponds to $r_1 = 1$.
(c) [1] Give the first four terms of the series solution found in part **(b)**.

Solution:

(a) Rewrite the equation in the standard form :

$$y'' - \frac{1}{2x}y' + \frac{1-x}{2x^2}y = 0.$$

Here $p(x) = -\dfrac{1}{2x}$, $xp(x) = -\dfrac{1}{2}$, $q(x) = \dfrac{1-x}{2x^2}$, $x^2q(x) = \dfrac{1-x}{2}$.

$p_0 = -\dfrac{1}{2}$, $q_0 = \dfrac{1}{2}$ $\Rightarrow$ $r^2 + (p_0 - 1)r + q_0 = r^2 - \dfrac{3}{2}r + \dfrac{1}{2} = 0$, *or* $2r^2 - 3r + 1 = 0$ is an indicial equation. The roots are $r_1 = 1$, $r_2 = \dfrac{1}{2}$.

(b) The solution $y(x)$ corresponding to $r_1 = 1$ has the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+1}, \text{ with } y' = \sum_{n=0}^{\infty}(n+1)\,a_n x^n \text{ and } y'' = \sum_{n=0}^{\infty} n(n+1)\,a_n x^{n-1}.$$

Substituting y, y' and y'' into the original equation yields

$$\sum_{n=0}^{\infty} 2n(n+1)a_n x^{n+1} - \sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

After combining the series for x^{n+1} the equation becomes

$$\sum_{n=0}^{\infty} n(2n+1)a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0. \qquad (*)$$

Notice that in the first series the first term, which corresponds to $n = 0$, is zero. So the series does not change if the summation starts with $n = 1$. Thus, if in that series we shift the index of summation $n \to n+1$, then the series becomes

$$\sum_{n=1}^{\infty} n(2n+1)a_n x^{n+1} = \sum_{n+1=1}^{\infty} (n+1)(2(n+1)+1)a_{n+1}x^{n+1+1} = \sum_{n=0}^{\infty}(n+1)(2n+3)a_{n+1}x^{n+2}.$$

Substituting the series above back to the equation $(*)$ yields

$$\sum_{n=0}^{\infty}(n+1)(2n+3)a_{n+1}x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0,$$

which can be combined into one series as

$$\sum_{n=0}^{\infty}\{(n+1)(2n+3)a_{n+1}-a_n\}\,x^{n+2}=0.$$

The above equation means that the series converges to 0 for all x near x_0. Therefore, all the coefficients in the series must be zero:

$$(n+1)(2n+3)a_{n+1}-a_n=0,$$

or

$$a_{n+1}=\frac{a_n}{(n+1)(2n+3)}.$$

Thus, we found the recurrence relation for the coefficients. Let us solve it.

$$n=0\ \Rightarrow a_1=\frac{1}{1\cdot 3}\,a_0;$$

$$n=1\ \Rightarrow a_2=\frac{a_1}{2\cdot 5}=\frac{1}{(1\cdot 2)\cdot(3\cdot 5)}a_0;$$

$$n=2\ \Rightarrow a_3=\frac{a_2}{3\cdot 7}=\frac{1}{(1\cdot 2\cdot 3)\cdot(3\cdot 5\cdot 7)}a_0;$$

$$n=3\ \Rightarrow a_4=\frac{a_3}{4\cdot 9}=\frac{1}{(1\cdot 2\cdot 3\cdot 4)\cdot(3\cdot 5\cdot 7\cdot 9)}a_0=\frac{1\cdot 2\cdot 4\cdot 6\cdot 8}{(1\cdot 2\cdot 3\cdot 4)\cdot(3\cdot 5\cdot 7\cdot 9\cdot 2\cdot 4\cdot 6\cdot 8)}a_0=$$
$$\frac{1\cdot 2\cdot(2\cdot 2)\cdot(3\cdot 2)\cdot(4\cdot 2)}{4!\cdot 9!}\,a_0=\frac{4!\cdot 2^4}{4!\cdot 9!}.$$

The pattern emerging for a_k is

$$a_k=\frac{2^k}{(2k+1)!}\,a_0.$$

Thus, $\displaystyle y=\sum_{n=0}^{\infty}a_n x^{n+1}=\sum_{n=0}^{\infty}\frac{2^n\,a_0}{(2n+1)!}\,x^{n+1}.$

(c) The first four terms of the solution:

$$y=\sum_{n=0}^{\infty}\frac{2^n\,a_0}{(2n+1)!}x^{n+1}\ =\ a_0(x+\frac{1}{3}x^2+\frac{1}{30}x^3+\frac{1}{630}\,x^4+...).$$