MATH 3705 AB
Test 2 - Solutions
February 2006

LAST NAME: ______________________________ **ID#:**________________

Questions 1-5 are multiple choice. Circle the correct answer. Only the answer will be marked.

1. [3 marks] The general solution of $x^2y'' + 2xy' + y = 0$ for $x \neq 0$ is

(a) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{\frac{\sqrt{3}}{2}}$ (b) $c_1|x|^{-\frac{1}{2}+\frac{\sqrt{3}}{2}} + c_2|x|^{-\frac{1}{2}-\frac{\sqrt{3}}{2}}$ (c) $|x|^{-\frac{1}{2}+\frac{\sqrt{3}}{2}}\left[c_1 + c_2 \ln|x|\right]$

(d) $|x|^{-\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right]$ (e) None of the above

2. [3 marks] The general solution of $4x^2y'' + 8xy' + y = 0$ for $x \neq 0$ is

(a) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$ (b) $|x|^{-\frac{1}{2}}\left[c_1 + c_2 \ln|x|\right]$ (c) $|x|^{-\frac{1}{2}}\left[c_1 \cos(\ln|x|) + c_2 \sin(\ln|x|)\right]$

(d) $|x|^{-\frac{1}{2}}\ c_1 \cos\left(-\frac{1}{2}\ln|x|\right) + c_2 \sin\left(-\frac{1}{2}\ln|x|\right)\Big]$ (e) None of the above

3. [2 marks] The general solution of $x^2y'' + xy' + (7x^2 - 1)y = 0$ for $x > 0$ is

(a) $c_1J_1(\sqrt{7}\,x) + c_2Y_1(\sqrt{7}\,x)$ (b) $c_1J_1(\sqrt{7}\,x) + c_2J_{-1}(\sqrt{7}\,x)$

(c) $c_1J_{\sqrt{7}}(x) + c_2J_{-\sqrt{7}}(x)$ (d) $c_1J_{\sqrt{7}}(x) + c_2Y_{\sqrt{7}}(x)$ (e) None of the above

4. [2 marks] The general solution of $x^2y'' + xy' + (4x^2 - 7)y = 0$ for $x > 0$ is

(a) $c_1J_2(\sqrt{7}\,x) + c_2J_{-2}(\sqrt{7}\,x)$ (b) $c_1J_2(\sqrt{7}\,x) + c_2Y_2(\sqrt{7}\,x)$

(c) $c_1J_{\sqrt{7}}(2x) + c_2J_{-\sqrt{7}}(2x)$ (d) $c_1J_{\sqrt{7}}(2x) + c_2Y_{\sqrt{7}}(2x)$ (e) None of the above

5. [2 marks] The general solution of $xy'' + y' + 3xy = 0$ for $x > 0$ is

(a) $c_1J_0(\sqrt{3}\,x) + c_2J_0(\sqrt{3}\,x)\ln(x)$ (b) $c_1J_0(\sqrt{3}\,x) + c_2Y_0(\sqrt{3}\,x)$

(c) $c_1J_{\sqrt{3}}(x) + c_2J_{-\sqrt{3}}(x)$ (d) $c_1J_{\sqrt{3}}(x) + c_2Y_{\sqrt{3}}(x)$ (e) None of the above

Answers: **d**, **b**, **a**, either **c** or **d**, **b**.

6. Consider the equation $2x^2y'' - xy' + (1-x)y = 0$.

 (a) [6 marks] Show that $x_0 = 0$ is a regular singular point and determine the indicial roots r_1 and r_2.

 (b) [12 marks] Find two linearly independent solutions y_1 and y_2, valid for $x > 0$.

Marking Scheme of 6(b): y_1 and y_2 may be found separately, or jointly.

$$\text{If jointly, then} \begin{cases} 6 & \text{marks for the recursion relation,} \\ 2 & \text{marks for finding } c_n \text{ explicitly,} \\ 1 & \text{marks for } a_n = c_n(r_1), \\ 1 & \text{marks for } b_n = c_n(r_2), \\ 1 & \text{mark for } y_1, \\ 1 & \text{marks for } y_2. \end{cases}$$

$$\text{If separately, then} \begin{cases} 3 & \text{marks for the recursion for } a_n, \\ 2 & \text{marks for the solution of } a_n, \\ 3 & \text{marks for the recursion for } b_n, \\ 2 & \text{marks for the solution of } b_n, \\ 1 & \text{mark for } y_1, \\ 1 & \text{mark for } y_2. \end{cases}$$

Solution:

(a) Rewrite the equation in the standard form :

$$y'' - \frac{1}{2x}y' + \frac{1-x}{2x^2}y = 0.$$

Here $p(x) = -\dfrac{1}{2x}$, $xp(x) = -\dfrac{1}{2}$, $q(x) = \dfrac{1-x}{2x^2}$, $x^2q(x) = \dfrac{1-x}{2}$.

Both $xp(x)$ and $x^2q(x)$ are analytic at $x_0 = 0$, so 0 is a regular singular point.

$p_0 = -\dfrac{1}{2}$, $q_0 = \dfrac{1}{2}$ $\Rightarrow$ $r^2 + (p_0 - 1)r + q_0 = r^2 - \dfrac{3}{2}r + \dfrac{1}{2} = 0$, *or* $2r^2 - 3r + 1 = 0$ is an indicial equation. The roots are $r_1 = 1$, $r_2 = \dfrac{1}{2}$.

(b) Two linearly independent solutions y_1 and y_2 may be found separately, or jointly. Below we do the computations jointly.

$$y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}, \quad y' = \sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(r)x^{n+r-2}.$$

Substituting y, y' and y'' into the original equation yields

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0.$$

Combine the series for x^{n+r} and for x^{n+r+1}:

$$\sum_{n=0}^{\infty}[2(n+r)(n+r-1)-(n+r)+1]c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0, \text{ or}$$

$$\sum_{n=0}^{\infty}[2(n+r)^2-3(n+r)+1]c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0. \qquad (*)$$

Notice that in the first series, when $n=0$, then the corresponding term is 0, because the coefficient $2r^2-3r+1$ equals zero for $r=1$ and $r=\frac{1}{2}$. So the series does not change if the summation starts with $n=1$. Thus, if we shift the index of summation $n \to n+1$ in that series, then it becomes

$$\sum_{n+1=1}^{\infty}[2(n+r+1)^2-3(n+r+1)+1]c_{n+1}x^{n+r+1} = \sum_{n=0}^{\infty}[2(n+r+1)^2-3(n+r+1)+1]c_{n+1}x^{n+r+1}.$$

Simplify the coefficient with $c_{n+1}x^{n+r+1}$:

$$2(n+r+1)^2-3(n+r+1)+1 = 2(n+r)^2+4(n+r)+2-3(n+r)-3+1 =$$
$$= 2(n+r)^2+(n+r) = (n+r)(2n+2r+1).$$

Then from equation $(*) \Rightarrow \sum_{n=0}^{\infty}(n+r)(2n+2r+1)c_{n+r}x^{n+r+1} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0,$

and the recursion relation is $\quad c_{n+1} = \dfrac{c_n}{(n+r)(2n+2r+1)}, \quad n \geq 0.$

$$n=0 \;\Rightarrow c_1 = \frac{c_0}{r(2r+1)};$$

$$n=1 \;\Rightarrow c_2 = \frac{c_1}{(1+r)(2r+3)} = \frac{c_0}{r(1+r)(2r+1)(2r+3)};$$

$$n=2 \;\Rightarrow c_3 = \frac{c_2}{(2+r)(2r+5)} = \frac{c_0}{r(1+r)(2+r)(2r+1)(2r+3)(2r+5)};$$

$$\cdots$$

$$n=n-1 \;\Rightarrow c_n = \frac{c_0}{r(1+r)(2+r)...(n-1+r)(2r+1)(2r+3)(2r+5)...(2r+2n-1)}, \quad n \geq 1.$$

Now $a_n = c_n(1) = \dfrac{c_0}{1\cdot 2\cdot ...n\cdot 3\cdot 5\cdot 7...(2n+1)} = \dfrac{2^n\cdot n!}{n!(2n+1)!}\,c_0 = \dfrac{2^n}{(2n+1)!}\,c_0.$

$$b_n = c_n(\frac{1}{2}) = \frac{c_0}{(\frac{1}{2})\cdot(\frac{3}{2})\cdot ...(\frac{2n-1}{2})\cdot 2\cdot 4\cdot 6...(2n)} = \frac{2^n}{(2n)!}\,c_0.$$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty}\frac{2^n\,c_0}{(2n+1)!}x^{n+1} \;=\; c_0(x+\frac{1}{3}x^2+\frac{1}{30}x^3+...).$$

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+1/2} = \sum_{n=0}^{\infty}\frac{2^n\,c_0}{(2n)!}x^{n+1/2} \;=\; c_0(x^{1/2}+x^{3/2}+\frac{1}{6}x^{5/2}+...).$$